Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 6

(Orientation, Riemannian manifolds)

due 28.11.2012

Exercise 1 5 points
For which $n \in \mathbb{N}$ is $\mathbb{R}P^n$ orientable, and for which not? Prove it.

Exercise 2 5 points
Prove that an $n$-dimensional manifold $M$ for which there exists an immersion $f: M \to \mathbb{R}^{n+1}$ is orientable if and only if there is a smooth nowhere-vanishing normal vector field along $(M, f)$, i.e. a map $N: M \to S^n$ such that $N(p) \perp d_pf(T_pM)$ for all $p \in M$.

Exercise 3 5 points
Consider $S^2 \subset \mathbb{R}^3$ with its induced metric. Show that the following maps are isometric immersions:

a) $f_1: S^2 \to \mathbb{R}^6$, \((x, y, z) \mapsto \left(\frac{1}{\sqrt{2}}x^2, \frac{1}{\sqrt{2}}y^2, \frac{1}{\sqrt{2}}z^2, xy, xz, yz\right)$,

b) $f_2: \mathbb{R}^2 \to \mathbb{R}^4$, \((u, v) \mapsto (\cos u, \sin u, \cos v, \sin v)$.

In how far can $f_1$ be considered as isometric immersion from the real projective plane $\mathbb{R}P^2$ to $\mathbb{R}P^5$, and $f_2$ as one from the 2-torus $T^2 = \mathbb{R}^2/2\pi \mathbb{Z}^2$. 