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http://www3.math.tu-berlin.de/geometrie/Lehre/WS12/DiffGeoII WS 12

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 7

(Integration, isometries)

due 05.12.2012

Exercise 1 **5 points**

Let $b > a > 0$. Denote by T^2 the torus in $\mathbb{R}^3$ obtained by rotating a circle of radius a in the xz–plane with center at $(b, 0, 0)$ around the z–axis. Determine the volume form $d\,\mathrm{vol}$ corresponding to the induced metric with respect to suitable coordinates and use it to determine $\mathrm{vol}\,(T^2)$.

Exercise 2 **6 points**

Let M be a smooth manifold and let G be a compact, connected Lie group that is a subgroup of $\mathrm{Diff}\,(M)$, the group of diffeomorphisms of M. Then there exists a Riemannian structure $\langle .,. \rangle$ on M such that the elements of G are isometries of $(M, \langle .,. \rangle)$. Hint: Use the Haar measure on G to average.

Exercise 3 **4 points**

a) Show that the metric $\langle X, Y \rangle := \frac{1}{2}\,\mathrm{trace}\left(\bar{X}^t Y\right)$ on $\mathrm{gl}\,(2, \mathbb{C})$ defines a Riemannian metric on $\mathrm{SU}\,(2)$.
b) Show that the left and the right multiplication by a constant g are isometries.
c) Show that $\mathrm{SU}\,(2, \mathbb{C})$ and the 3–sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ (with induced metric) are isometric. Hint: $\mathrm{SU}\,(2, \mathbb{C}) = \left\{ \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix} \right) \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$.