Exercise 1 5 points
Let $b > a > 0$. Denote by $T^2$ the torus in $\mathbb{R}^3$ obtained by rotating a circle of radius $a$ in the $xz$–plane with center at $(b, 0, 0)$ around the $z$–axis. Determine the volume form $d\text{vol}$ corresponding to the induced metric with respect to suitable coordinates and use it to determine $\text{vol} (T^2)$.

Exercise 2 6 points
Let $M$ be a smooth manifold and let $G$ be a compact, connected Lie group that is a subgroup of $\text{Diff} (M)$, the group of diffeomorphisms of $M$. Then there exists a Riemannian structure $\langle \cdot, \cdot \rangle$ on $M$ such that the elements of $G$ are isometries of $(M, \langle \cdot, \cdot \rangle)$. Hint: Use the Haar measure on $G$ to average.

Exercise 3 4 points
a) Show that the metric $\langle X, Y \rangle := \frac{1}{2} \text{trace} (\bar{X} Y)$ on $\text{gl} (2, \mathbb{C})$ defines a Riemannian metric on $\text{SU} (2)$.

b) Show that the left and the right multiplication by a constant $g$ are isometries.

c) Show that $\text{SU} (2, \mathbb{C})$ and the 3–sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ (with induced metric) are isometric. Hint: $\text{SU} (2, \mathbb{C}) = \{ ( \begin{array}{cc} a & b \\ -\bar{b} & \bar{a} \end{array} ) \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \}$. 

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