Technische Universität Berlin
Institut für Mathematik
Springborn / Knöppel

http://www3.math.tu-berlin.de/geometrie/Lehre/WS13/ComplexAnalysisII WS 13

Complex Analysis II - Exercise Sheet 11

- Abelian Differentials -

due 29.1.2014

Exercise 1 — 5 points

Given the following data on a compact Riemann surface $\mathcal{R}$:

- N points $p_1, \dots, p_N \in \mathcal{R}$,
- local coordinates $z_1, \dots, z_N$ around these points with $z_i(P_i) = 0$,
- complex numbers $a_{-1}^{(k)}, \dots, a_{-n_k}^{(k)}$ for $k = 1, \dots, N$ satisfying $\sum_{k=1}^{N} a_{-1}^{(k)} = 0$,
- complex numbers $A_1, \dots, A_g$.

Show that here exists a unique Abelian differential ω with series expansion

$$\omega = \left(a_{-n_k}^{(k)} z_k^{-n_k} + \dots + a_{-1}^{(k)} z_k^{-1} + \mathcal{O}(1)\right) dz_k$$

near the points p_k, no poles elsewhere, and the A-periods $\int_{a_j} \omega = A_j$.

Hint: Use the existence of normalized Abelian differentials $\omega_1, \dots, \omega_g$ of the first kind, $\omega_{P,n}^{II}$ of the second kind, and $\omega_{P,Q}^{III}$ of the third kind (existence-theorem I-III), as well as the fact that a holomorphic differential with vanishing A-periods is identically zero.

Exercise 2 — 5 points

Let $\mathcal{R}$ be a compact Riemann surface. Let f denote a meromorphic function, and let ω be an Abelian differential on $\mathcal{R}$. Show:

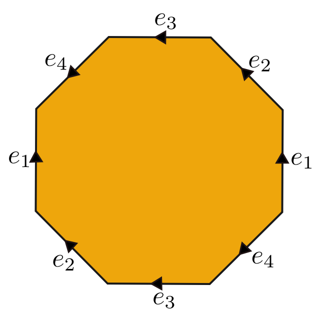
a) If f has a pole of order n at a point $p \in \mathcal{R}$, then there is a chart z around p with $z(p) = 0$ such that $f(z) = \frac{1}{z^n}$.

b) If ω has a pole of order $n \geq 2$ at a point $p \in \mathcal{R}$ such that $\mathrm{Res}_p(\omega) = 0$, then there is a chart z around p with $z(p) = 0$ such that $\omega = \frac{1}{z^n}\, dz$.

c) If ω has a pole of order 1 at a point $p \in \mathcal{R}$ with $\mathrm{Res}_p(\omega) = r$, then there is a chart z around p with $z(p) = 0$ such that $\omega = \frac{r}{z}\, dz$.

Exercise 3 — 5 points

Let $G \subset \mathbb{C}$ denote a regular octagon with edges of length 1 parallel to either the x-axis, y-axis or one of their angle bisectors. If we identify opposite edges by the corresponding translation we obtain a compact Riemann surface $\mathcal{R}$ of genus 2 with the conformal structure of a polyhedral surface.



a) Show that dz induces a holomorphic differential ω on $\mathcal{R}$ and determine its zeros.

b) Calculate the periods of ω with respect to a canonical homology basis.