Shadow Problem

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Preliminaries

What is a convex surface?

Definition

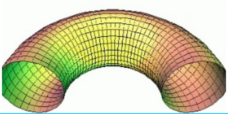
A set A in a real vector space is convex if for any $a, b \in A$ we have $c \in A$ for every point $c = (1-\lambda)a + \lambda b$ with $\lambda \in [0,1]$

Observation

- With this definition $\mathbb{S}^2$ is non-convex is $\mathbb{R}^3$.
- When dealing with two dimensional surfaces embedded in $\mathbb{R}^3$, the convexity property is redefined: (Pogorelov 1969)

Definition

A smooth oriented surface M in $\mathbb{R}^3$ is a **convex surface** if there is a convex body A of $\mathbb{R}^3$ such that M is exactly on the boundary of A and the normal of each point p of M points toward the exterior of A. (i.e. Let $f : M \to \mathbb{R}^3$ is a smooth immersion into $\mathbb{R}^3$ then f is a **convex embedding**, if f maps M homeomorphically to the boundary of a convex body).



Introduction

- Artists have long understood the importance of shadows for generating an impression of three-dimensionality in paintings.
- Shadows provide a perceptually salient information about the shapes of objects and the direction of illumination in a scene.
- Let M be a closed oriented 2-dimensional manifold, and $f : M \to \mathbb{R}^3$ be a smooth immersion and $n : M \to \mathbb{S}^2$ a continuous map s.t. $n(p)$ is a unit vector orthogonal to M at each p and $u \in \mathbb{S}^2$ be a unit vector.
- Suppose that M is illuminated by parallel rays of light flowing in the direction of u

Definition

For each $u \in \mathbb{S}^2$ we define the **shadow function** $\sigma_u : M \to \mathbb{R}$ by $\sigma_u(p) =< n(p), u >$.

Definition

A **shadow** set on M is the set of points in M not reached by the rays of light, given by

$$S_u = \{p \in M : \sigma_u > 0\}.$$

Shadow Problem

- If a convex surface, such as an egg shell, is illuminated from any given direction, then the corresponding shadow cast on the surface forms a connected subset.
- The shadow Conjecture, studied and stated by Wente, asks whether the converse of this is true.

Shadow Conjecture

Let M be an oriented compact surface immersed in $\mathbb{R}^3$. Suppose that for every unit vector $u \in \mathbb{S}^2$, the corresponding shadow, S_u is **simply connected** then M is convex.

Observation

The compactness assumption cannot be removed. Suppose for instance that M is a hyperbolic paraboloid $z = xy$ which is not bounded. Then all the shadows of M are simply connected, although M is not a convex surface.
Let $H_u = \{x \in S^2 :< x, u >> 0\}$. The Gauss map of M, n, is a homeomorphism into $H_{(0,0,1)}$ then $S_u = \{p \in M :< n(p), u >> 0\} = n^{-1}(H_u) = n^{-1}(H_u \cap H_{(0,0,1)})$. Thus since $H_u \cap H_{(0,0,1)}$ is simply connected, it follows that S_u is also simply connected.

Shadow Boundary

Note

The **shadow boundary** is ∂S_u in the direction of u. The boundaries are formed anywhere that the light ray is tangent to the surface.

One of the basic properties of this curve is that they are everywhere smooth:

Proposition

For almost all $u \in \mathbb{S}^2$ *the set* $\{p \in M : \sigma_u(p) = 0\}$ *is a smooth curve in* M. *Thus, for these* u *we have that* $\partial S_u = \{p \in M : \sigma_u(p) = 0\}$ *and so* ∂S_u *is a smooth curve for almost all* $u \in \mathbb{S}^2$.

Proof.

Let $T_p(M)$ be the tangent plane of M at p,
$U(M) = \{(p,u) : p \in M, u \in T_p(M), ||u|| = 1\}$ the tangent bundle of M and
$\tau : U(M) \to \mathbb{S}^2$ given by $\tau(p,u) = u \in \mathbb{S}^2$. By Sard's Theorem almost every $u \in \mathbb{S}^2$ is a regular value of τ and for such u, $\tau^{-1}(u)$ is a smooth curve in $U(M)$. If $(p,u) \in U(M)$ and $u_1 \in T_p(M)$ is a unit vector with $< u, u_1 >= 0$ then $c(t) = (p, cos(t)u + sin(t)u_1)$ parametrizes $U_p(M)$ and $c(0) = (p,u)$. Also (let τ_* denote the differential)

$$\tau_*[c'(0)] = \frac{d}{dt}\tau(p, cos(t)u + sin(t)u_1)|_{t=0} = \frac{d}{dt}(cos(t)u + sin(t)u_1)|_{t=0} = u_1 \neq 0.$$

Proof.

If u is a regular value of τ and $(p, u) \in \tau^{-1}[u]$ then the tangent space to $\tau^{-1}[u]$ is $T_{(p,u)}(t^{-1}(u)) = \{X \in T_{(p,u)}[U(M)] : \tau_* X = 0\}$. As $\tau_*(c'(0)) \neq 0$ implies that the curve $\tau^{-1}[u]$ is never tangent to $U_p(M)$.
Let $\pi : U(M) \to M$ be the natural projection and u a regular value of τ. By definitions, $\pi|_{\tau^{-1}(u)} : \tau^{-1}(u) \to \mathbb{S}^2$ is injective. Since $\tau^{-1}[u]$ is never tangent to $U_p(M) = \pi^{-1}[p]$ the map $\pi|_{\tau^{-1}(u)}$ is an immersion.
As $\tau^{-1}(u)$ is compact this implies that the projection $\pi[\tau^{-1}[u]]$ is a smooth embedded curve in M. Moreover,

$$\pi[\tau^{-1}(u)] = \{p \in M : u \in T_p(M)\} = \{p \in M :< u, n(p) >= 0\} = \{p \in M : \sigma_u(p) = 0\}$$

□

Surface diffeomorphic to a sphere

Corollary

Let M be a compact oriented surface immersed in $\mathbb{R}^3$ so that all the shadow sets S_e are simply connected then M is diffeomorphic to the sphere.

Proof.

Let u be a regular value of τ then ∂S_u is a smooth curve. As S_u is simply connected the closure $\overline{S_u}$ will be diffeomorphic to a closed disk. The boundary of S_{-u} is the same as the boundary of S_u and so $\overline{S_{-u}}$ is also a closed disk. The surface M is the disjoint union of $S_u, S_{-u}, \partial S_u$. Thus M is a pair of disks glued together along their boundaries so M is a sphere. □

Shadow Problem

Theorem

Let $M \subset \mathbb{R}^3$ be a compact oriented surface and $f : M \to \mathbb{R}^3$ a smooth immersion. f is a convex embedding if and only if, for every $u \in \mathbb{S}^2$, S_u is simply connected.

Proof.

If f is a convex embedding then let $u \in \mathbb{S}^2$ and Π be a plane perpendicular to u and let $\pi : \mathbb{R}^3 \to \Pi$ be the orthogonal projection. Then $D = \pi(f(M))$ is a convex subset of Π with interior points. In particular int(D) is homeomorphic to an open disk. Since $f(M)$ is convex and by definition $\sigma_u(p) > 0$ for all $p \in S_u$ it can be verified that $f(S_u)$ is a graph over int(D). Thus $\pi \circ f : S_u \to int(D)$ is a homeomorphism. □

Shadow Problem

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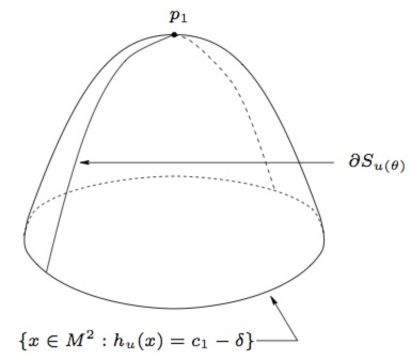
Proof.

Outline of the proof.
The proof is by contradiction and is organized basically in 3 steps:
Suppose that M satisfies the hypothesis but is not convex, Then there exists a unit vector $v \in \mathbb{S}^2$ such that the corresponding height function $h_v : M \to \mathbb{R}$ is defined by $h_v(p) =< p, v >$ which has at least three critical points (two local maximums).
For each $v \in \mathbb{S}^2$ let $v^\perp \cap \mathbb{S}^2$ be the great circle of $\mathbb{S}^2$ orthogonal to v. Then as v varies over $\mathbb{S}^2$ the circles $v^\perp \cap \mathbb{S}^2$ will cover a set F of positive measure in $\mathbb{S}^2$. By previous proposition there is a $u_0 \in F$ s.t. ∂S_{u_0} is a smooth curve in M. Now we choose $v \in S^2$ st $u_0 \in v^\perp \cap \mathbb{S}^2$. Then we have a pair $v \perp u_0$ and the height function h_v with at least two local maximums.
By rotation we can assume that v is in the direction of the positive $z-$axis and u_0 in the direction of positive $x-$axis. The height function h_u is then the restriction of the z coordinate to M. Let $u_0(\theta) = (cos\theta, sin\theta, 0)$, so $u_0(0) = u_0$. Let p_1, p_2 two local maximums and p_3 a local minima. Since $v \perp u_0$ for all θ, $p_i \in S_{u_0(\theta)}$. It is then proved that the Gauss curvature is non-zero at all critical points and thus the Gauss map of M is a local diffeomorphism near each of these points.

Continuation

Proof.



Near the local maxima p_1 a plane parallel to the tangent $T_{p_1}(M)$ and a little below p_1 will cut off a disk D_1 in M so that the restriction of the Gauss map to D_1 is a diffeomorphism onto its image, which implies that $D_1 \cap \partial S_{u_0\theta}$ is a smooth curve through the point p_1. We can construct similar Disks for other points so that $g|_{D_i} : D_i \longrightarrow S^2$ is a diffeomorphism. As $D_i \cap \partial S_{u_0\theta}$ is smooth for all θ and we are assuming that each $S_{u_0(\theta)}$ is simple connected we see that $(p_1, p_2, p_3, S_{u_0(\theta)})$ is a nicely placed triple.

□

Last step

Proof.

It is then proved that for each θ the triple determines a permutation $\sigma(p_1, p_2, p_3, S_{u_0(\theta)})$ of the set $\{p_1, p_2, p_3\}$. We can assume that when $\theta = 0$ $\sigma(p_1, p_2, p_3, S_{u_0(\theta)}) = (p_1 p_2 p_3)$ and can is furthered proved that $\theta \to \sigma(p_1, p_2, p_3, S_{u_0(\theta)})$ is locally constant. As the real numbers are connected then $\sigma(p_1, p_2, p_3, S_{u_0(\theta)}) = (p_1 p_2 p_3)$ for all θ.
Recall when $\theta = 0$, $\partial S_{u_0(0)}$ is a smooth boundary. It is then proved that the permutation $\sigma(p_1, p_2, p_3, S_{u_0(\theta)})$ is just the cyclic permutation induced on $\{p_1, p_2, p_3\}$ by the boundary orientation of $\partial S_{u_0(0)}$ .But $u_0(\pi) = -u_0(0)$ and $S_{u_0(\pi)} = M - \overline{S_{u_0(0)}}$.
Therefore as sets $\partial S_{u_0(0)} = \partial S_{u_0(\pi)}$ but as the inward normal determined by $S_{u_0(0)}$ is the outward normal determined by $S_{u_0(\pi)}$ the boundary orientation $\partial S_{u_0(0)}$ determined by $S_{u_0(\pi)}$ is the opposite of the one determined by $S_{u_0(0)}$. Thus if $\sigma(p_1, p_2, p_3, S_{u_0(0)}) = (p_1 p_2 p_3)$ then $\sigma(p_1, p_2, p_3, S_{u_0(\pi)}) = (p_1 p_3 p_2)$ which is a contradiction.

□

Why simple connected?

Connectedness of the shadows in general is not strong enough to ensure a convex surface.

Theorem

There exists a smooth embedding of the torus $f : S^1 \times S^1 \to \mathbb{R}^3$ *such that for all* $u \in S^2$, S_u *is connected.*

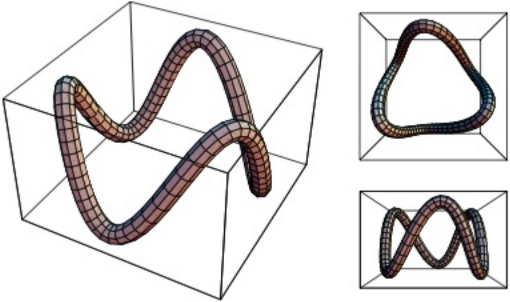


Figure: A tube about a curve that has no two parallel tangent lines. Such a tube has all its shadow sets connected.