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http://www3.math.tu-berlin.de/geometrie/Lehre/WS14/DiffGeoII WS 14

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 1

(Topological manifolds)

due 30.10.2014

Exercise 1 — 5 points

Let X be a topological space and $x \in \mathrm{X}$. A *neighborhood of* x is a subset V of X that contains an open set U containing x, i.e. $x \in U \subset V$. Let $n \geq 0$. Show that the following statements are equivalent:

i) There is a neighborhood of x which is homeomorphic to $\mathbb{R}^n$.
ii) There is a neighborhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$.

Exercise 2 — 5 points

Let M and N be topological manifolds of dimension m and n, resp. Show that their Cartesian product $\mathrm{M} \times \mathrm{N}$ is a manifold of dimension $m + n$.

Exercise 3 — 5 points

Set $\mathrm{X} := \mathbb{R}^{n+1} \setminus \{0\}$. The *real projective space* $\mathbb{RP}^n$ is then defined as the quotient space of X with respect to the following equivalence relation:

$$x \sim y :\Longleftrightarrow x = \lambda y, \quad \lambda \in \mathbb{R}.$$

The *canonical projection* $\pi \colon X \to \mathbb{RP}^n$, maps the elements of X to its equivalence classes: $\pi(x) := [x]$. With the *quotient topology*, i.e. the topology defined by

$$U \subset \mathbb{RP}^n \text{ open} :\Longleftrightarrow \pi^{-1}(U) \subset \mathrm{X} \text{ open},$$

it becomes a topological manifold of dimension n. Show that $\mathbb{RP}^n$ is compact.

Hint: Show that the restriction of the canonical projection π to $\mathbb{S}^n$ is surjective.