Exercise Sheet 1
(Topological manifolds)
due 30.10.2014

Exercise 1 5 points
Let $X$ be a topological space and $x \in X$. A *neighborhood of $x$* is a subset $V$ of $X$ that contains an open set $U$ containing $x$, i.e. $x \in U \subset V$. Let $n \geq 0$. Show that the following statements are equivalent:

i) There is a neighborhood of $x$ which is homeomorphic to $\mathbb{R}^n$.

ii) There is a neighborhood of $x$ which is homeomorphic to an open subset of $\mathbb{R}^n$.

Exercise 2 5 points
Let $M$ and $N$ be topological manifolds of dimension $m$ and $n$, resp. Show that their Cartesian product $M \times N$ is a manifold of dimension $m + n$.

Exercise 3 5 points
Set $X := \mathbb{R}^{n+1} \setminus \{0\}$. The *real projective space* $\mathbb{P}^n$ is then defined as the quotient space of $X$ with respect to the following equivalence relation:

$$x \sim y :\iff x = \lambda y, \quad \lambda \in \mathbb{R}. $$

The *canonical projection* $\pi : X \to \mathbb{P}^n$, maps the elements of $X$ to its equivalence classes: $\pi(x) := [x]$. With the *quotient topology*, i.e. the topology defined by

$$U \subset \mathbb{P}^n \text{ open} :\iff \pi^{-1}(U) \subset X \text{ open},$$

it becomes a topological manifold of dimension $n$. Show that $\mathbb{P}^n$ is compact.

**Hint:** Show that the restriction of the canonical projection $\pi$ to $S^n$ is surjective.