Exercise Sheet 2

(Smooth manifolds)

due 6.11.2014

Exercise 1 3 points
Let M be a topological manifold. Show that M has an exhaustion by compact sets, i.e. there exists a sequence $K_1 \subset K_2 \subset K_3 \subset \cdots \subset M$ of compact sets such that $K_i \subset \mathring{K}_{i+1}$ and $\bigcup_{i=1}^{\infty} K_i = M$.

Exercise 2 4 points
Let M and N be smooth manifolds of dimension $m$ and $n$, respectively. The topological product $M \times N$ is a topological manifold of dimension $m + n$. Let $\{(U_i, \varphi_i)\}_{i \in I}$ a smooth atlas for M and $\{(V_j, \psi_j)\}_{j \in J}$ a smooth atlas for N. Show that $\{(U_i \times V_j, \varphi_i \times \psi_j)\}_{(i,j) \in I \times J}$ is a smooth atlas for $M \times N$. Here $\varphi_i \times \psi_j : U_i \times V_j \to \varphi_i(U_i) \times \psi_j(V_j)$ with $(\varphi_i \times \psi_j)(x,y) = (\varphi_i(x), \psi_j(y))$.

Exercise 3 4 points
Let $n \geq 0$ and let $J = \{0, \ldots, n\}$. We define $\left( U_j^\pm, \varphi_j^\pm \right)_{j \in J}$ by

$U_j^\pm := \{ x = (x_0, \ldots, x_n) \in \mathbb{S}^n \mid \pm x_j > 0 \}$,

$\varphi_j^\pm : U_j^\pm \to \mathbb{R}^n$, $\varphi_j^\pm(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

Show that $\left( U_j^\pm, \varphi_j^\pm \right)_{j \in J}$ is a $C^\infty$-atlas for $\mathbb{S}^n$.

Exercise 4 4 points
$M := \{ P \in \text{Mat}(3 \times 3, \mathbb{R}) \mid P^2 = P, P^* = P, \text{tr}(P) = 1 \} \subset \text{Sym}(3) \cong \mathbb{R}^6$. Show that M is a submanifold diffeomorphic to $\mathbb{R}P^2$. 