Technische Universität Berlin
Institut für Mathematik
Pinkall / Knöppel
http://www3.math.tu-berlin.de/geometrie/Lehre/WS14/DiffGeoII WS 14

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 3

(Tangent bundle, embedding)

due 13.11.2014

Exercise 1 **6 points**

Let M be a smooth manifold of dimension n. For each chart $\varphi\colon U \to \mathbb{R}^n$ we define $\mathrm{T}U := \bigcup_{p\in U} \mathrm{T}_p\mathrm{M} \subset \mathrm{TM}$ and a map $\Phi\colon \mathrm{T}U \to \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ by

$$\mathrm{T}_p\mathrm{M} \ni X_p \mapsto (\varphi(p), X_p\varphi) \in \mathbb{R}^{2n}.$$

Show:

i) Φ is bijective.

ii) If $\Psi\colon \mathrm{T}V \to \psi(V) \times \mathbb{R}^n$ is another such map corresponding to a chart $\psi\colon V \to \mathbb{R}^n$ and $W := U \cap V \neq \emptyset$, then

$$\Psi \circ \Phi^{-1}\colon \Phi(\mathrm{T}W) \to \Psi(\mathrm{T}W)$$

is a diffeomorphism.

Remark: If we use the topology induced by these maps, TM becomes a smooth manifold of dimension $2n$ – the tangent bundle.

Exercise 2 **5 points**

Prove that the tangent bundle of a product of manifolds is diffeomorphic to the product of the tangent bundles of the manifolds. Deduce that the tangent bundle of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$.

Exercise 3 **4 points**

Let $f\colon \mathrm{N} \to \mathrm{M}$ be a smooth immersion. Prove: If f is moreover a *topological embedding*, i.e. its restriction $f\colon \mathrm{N} \to f(\mathrm{N})$ is a homoeomorphism between N and $f(\mathrm{N})$ (with its subspace topology), then $f(\mathrm{N})$ is a smooth submanifold of M.