Exercise Sheet 5
(Flows, vector fields, tensors)
due 27.11.2014

Exercise 1
5 points
a) Sketch for each $n \geq 0$ a flow on $S^1$ with exactly $n$ fixed points.
b) Construct a flow on $S^{2n+1}$ with no fixed point.
c) Sketch a flow on $S^2$ with exactly two fixed points, which admits exactly one closed orbit.

Exercise 2
5 points
On $S^2 = \{ x = (x_0, x_1, x_2) \mid \| x \| = 1 \} \subset \mathbb{R}^3$ we consider coordinates given by the stereographic projection from the north pole $N = (1, 0, 0)$:

$$y_1 = \frac{x_1}{1-x_0}, \quad y_2 = \frac{x_2}{1-x_0}.$$

Let the vector fields $X$ and $Y$ on $S^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}; \quad Y = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole $S = (-1, 0, 0)$.

Exercise 3
5 points
Let $V$ and $W$ be finite-dimensional real vector spaces. The tensor product of $V$ and $W$ was explicitly defined as the following vector space:

$$V \otimes W := \{ \beta : V^* \times W^* \to \mathbb{R} \mid \beta \text{ bilinear} \}.$$

Clearly, $\otimes : V \times W \to V \otimes W$, $(v \otimes w)(\omega, \eta) := \omega(v)\eta(w)$ is bilinear. Show:
a) If $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ are bases of $V$ and $W$ (resp.), then $\{v_i \otimes w_j\}_{(i,j) \in I \times J}$ is a basis of $V \otimes W$.
b) For each bilinear map $\mu : V \times W \to X$ into a real vector space $X$ there is a unique linear map $\lambda : V \otimes W \to X$ such that $\mu = \lambda \circ \otimes$. 