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http://www3.math.tu-berlin.de/geometrie/Lehre/WS14/DiffGeoII WS 14

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 5

(Flows, vector fields, tensors)

due 27.11.2014

Exercise 1 **5 points**

a) Sketch for each $n \geq 0$ a flow on $\mathbb{S}^1$ with exactly n fixed points.
b) Construct a flow on $\mathbb{S}^{2n+1}$ with no fixed point.
c) Sketch a flow on $\mathbb{S}^2$ with exactly two fixed points, which admits exactly one closed orbit.

Exercise 2 **5 points**

On $\mathbb{S}^2 = \{x = (x_0, x_1, x_2) \mid \|x\| = 1\} \subset \mathbb{R}^3$ we consider coordinates given by the stereographic projection from the north pole $N = (1, 0, 0)$:

$$y_1 = \tfrac{x_1}{1-x_0}, \quad y_2 = \tfrac{x_2}{1-x_0}.$$

Let the vector fields X and Y on $\mathbb{S}^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2 \tfrac{\partial}{\partial y_1} - y_1 \tfrac{\partial}{\partial y_2}, \qquad Y = y_1 \tfrac{\partial}{\partial y_1} + y_2 \tfrac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole $S = (-1, 0, 0)$.

Exercise 3 **5 points**

Let V and W be finite-dimensional real vector spaces. The tensor product of V and W was explicitly defined as the following vector space:

$$\mathrm{V} \otimes \mathrm{W} := \big\{\beta \colon \mathrm{V}^* \times \mathrm{W}^* \to \mathbb{R} \mid \beta \textit{ bilinear}\big\}.$$

Clearly, $\otimes \colon \mathrm{V} \times \mathrm{W} \to \mathrm{V} \otimes \mathrm{W}$, $(v \otimes w)(\omega, \eta) := \omega(v)\eta(w)$ is bilinear. Show:

a) If $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ are bases of V and W (resp.), then $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$ is a basis of $\mathrm{V} \otimes \mathrm{W}$.
b) For each bilinear map $\mu \colon \mathrm{V} \times \mathrm{W} \to \mathrm{X}$ into a real vector space X there is a unique linear map $\lambda \colon \mathrm{V} \otimes \mathrm{W} \to \mathrm{X}$ such that $\mu = \lambda \circ \otimes$.