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http://www3.math.tu-berlin.de/geometrie/Lehre/WS14/DiffGeoII WS 14

# Differential Geometry II: Analysis and Geometry on Manifolds

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## Exercise Sheet 7

(Cartan's magic formula, integration)

due 11.12.2014

**Exercise 1** **5 points**

Let M be a smooth manifold and $X \in \Gamma(\mathrm{TM})$. Define $i_X\colon \Omega^k(\mathrm{M}) \to \Omega^{k-1}(\mathrm{M})$ and $\mathcal{L}_X\colon \Omega^k(\mathrm{M}) \to \Omega^k(\mathrm{M})$ as follows: If $\omega \in \Omega^k(\mathrm{M})$, then for $X_i \in \Gamma(\mathrm{TM})$

$$(i_X\omega)(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}),$$

$$(\mathcal{L}_X\omega)(X_1, \dots, X_k) := X(\omega(X_1, \dots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \dots, [X, X_i], \dots, X_k).$$

Show that for $k \geq 1$

$$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega).$$

**Exercise 2** **5 points**

Let $\mathrm{M} \subset \mathbb{R}^n$ be an $n$-dimensional submanifold with boundary with outward pointing normal vector field $N\colon \partial\mathrm{M} \to \mathrm{S}^{n-1} \subset \mathbb{R}^n$. Then the volume form of M is given by $\omega_{\mathrm{M}} = \det$ and the induced volume form on $\partial M$ is $\omega_{\partial\mathrm{M}} = i_N\omega_{\mathrm{M}}$. Let $X \in \Gamma(\mathrm{T}\mathbb{R}^n)$. Show that

$$\int_{\mathrm{M}} \operatorname{div}(X)\omega_{\mathrm{M}} = \int_{\partial\mathrm{M}} i_X\omega_{\mathrm{M}} = \int_{\partial\mathrm{M}} \langle X, N\rangle\omega_{\partial\mathrm{M}}.$$

**Exercise 3** **5 points**

For which $n \in \mathbb{N}$ is $\mathbb{RP}^n$ orientable, and for which not? Prove it.