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http://www3.math.tu-berlin.de/geometrie/Lehre/WS14/DiffGeoII WS 14

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 9

(Levi-Civita connection, geodesics)

due 8.1.2015

Exercise 1 **5 points**

Let (M, g) be a Riemannian manifold and $\tilde{g} = e^{2u} g$ for some smooth function $u\colon \mathrm{M} \to \mathbb{R}$. Show that between the corresponding Riemannian connections the following relation holds:

$$\tilde{\nabla}_X Y = \nabla_X Y + du(X)Y + du(Y)X - g(X,Y)\operatorname{grad} u.$$

Here $\operatorname{grad} u \in \Gamma(\mathrm{TM})$ is the vector field uniquely determined by the condition $du(X) = g(\operatorname{grad} u, X)$ for all $X \in \Gamma(\mathrm{TM})$.

Exercise 2 **5 points**

Let M be a Riemannian manifold, $\gamma\colon I \to \mathrm{M}$ a curve, which is parametrized with constant speed, and $f\colon \mathrm{M} \to \mathrm{M}$ an isometry, which fixes γ, i.e. $f \circ \gamma = \gamma$. Further, let

$$\ker\big(\mathrm{Id} - d_{\gamma(t)} f\big) = \mathbb{R}\dot{\gamma}(t), \ \textit{for all } t.$$

Then γ is a geodesic.

Exercise 3 **5 points**

The Poincaré half-plane model of the n-dimensional hyperbolic space is the half-plane $\mathrm{H}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ equipped with the metric $\frac{1}{x_n^2}\langle .,. \rangle$, where $\langle .,. \rangle$ denotes the Euclidean metric of $\mathbb{R}^n$.

a) Show that $\mathrm{B} \to \mathrm{H}^n_+$, $(\tilde{x}, x_n) \mapsto 2\frac{(\tilde{x}, 1-x_n)}{|(\tilde{x}, 1-x_n)|^2} - e_n$ is an isometry, where $\mathrm{B} = \{x \in \mathbb{R}^n \mid |x| < 1\}$ is equipped with the metric g of the Poincaré ball-model, i.e. $g_x(.,.) = \frac{4\langle .,. \rangle}{(1-|x|^2)^2}$.

b) Show that the maps $z \mapsto \frac{az+b}{cz+d}$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ define orientation preserving isometries of $\mathrm{H}^2_+ \subset \mathbb{C}$, and that this defines a group isomorphism between the isometries of H^2_+ and $\mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$.

c) Show that all geodesics of H^2_+ are either rays or semi-circular arcs, which are perpendicular to the real axis $\mathbb{R} \times \{0\}$.