Technische Universität Berlin
Institut für Mathematik
Pinkall / Knöppel
http://www3.math.tu-berlin.de/geometrie/Lehre/WS14/DiffGeoII WS 14

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 11

(Exponential map, parallel transport)

due 22.1.2015

Exercise 1 — 5 points

Let (M, g) be a Riemannian manifold of dimension n. Show that at each point $p \in \mathrm{M}$ there is a local coordinate $\varphi = (x_1, \ldots, x_n))$ at p such that

$$g\big(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\big)\Big|_p = \delta_{ij}, \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\Big|_p = 0.$$

Exercise 2 — 5 points

a) Is there a Riemannian manifold (M, g) which has finite diameter (i.e. there is an m such that all points $p, q \in \mathrm{M}$ have distance $d(p,q) < m$) and there is a geodesic of infinite length without self-intersections?

b) Find an example for a Riemannian manifold diffeomorphic to $\mathbb{R}^n$ but which has no geodesic of infinite length.

Exercise 3 — 5 points

Let γ be a curve in a smooth manifold M and $\gamma(t_0)$ a point on it. The mapping $P_{\gamma(t),\gamma(t_0)} \colon \mathrm{T}_{\gamma(t_0)}\mathrm{M} \to \mathrm{T}_{\gamma(t)}M$ defined by $P_{\gamma(t),\gamma(t_0)} X_{\gamma(t_0)} = X_{\gamma(t)}$, where $X_{\gamma(t)}$ is the unique extension of $X_{\gamma(t_0)}$ to a parallel vector field along γ, is called the parallel transport from $X_{\gamma(t_0)}$ to $X_{\gamma(t)}$. Show:

a) The parallel transport is a linear isomorphism, and if $X(t) = X_{\gamma(t)}$ is a vector field along γ, then

$$\tfrac{D}{dt} X\Big|_{t=t_0} = \lim_{t \to t_0} \tfrac{1}{t - t_0}\big(P_{\gamma(t_0),\gamma(t)} X(t) - X(t_0)\big).$$

b) If M is an oriented Riemannian manifold with Riemannian connection ∇, then $P_{\gamma(t),\gamma(t_0)}$ is an isometry preserving orientation.