Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 11
(Exponential map, parallel transport)
due 22.1.2015

Exercise 1 5 points
Let \((M, g)\) be a Riemannian manifold of dimension \(n\). Show that at each point \(p \in M\) there is a local coordinate \(\varphi = (x_1, \ldots, x_n)\) at \(p\) such that

\[
g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\bigg|_p = \delta_{ij}, \quad \nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \bigg|_p = 0.
\]

Exercise 2 5 points
a) Is there a Riemannian manifold \((M, g)\) which has finite diameter (i.e. there is an \(m\) such that all points \(p, q \in M\) have distance \(d(p, q) < m\)) and there is a geodesic of infinite length without self-intersections?
b) Find an example for a Riemannian manifold diffeomorphic to \(\mathbb{R}^n\) but which has no geodesic of infinite length.

Exercise 3 5 points
Let \(\gamma\) be a curve in a smooth manifold \(M\) and \(\gamma(t_0)\) a point on it. The mapping \(P_{\gamma(t), \gamma(t_0)} : T_{\gamma(t_0)} M \to T_{\gamma(t)} M\) defined by \(P_{\gamma(t), \gamma(t_0)} X_{\gamma(t_0)} = X_{\gamma(t)}\), where \(X_{\gamma(t)}\) is the unique extension of \(X_{\gamma(t_0)}\) to a parallel vector field along \(\gamma\), is called the parallel transport from \(X_{\gamma(t_0)}\) to \(X_{\gamma(t)}\). Show:

a) The parallel transport is a linear isomorphism, and if \(X(t) = X_{\gamma(t)}\) is a vector field along \(\gamma\), then

\[
\frac{D}{dt} X\bigg|_{t=t_0} = \lim_{t \to t_0} \frac{1}{t-t_0} \left( P_{\gamma(t), \gamma(t_0)} X(t) - X(t_0) \right).
\]

b) If \(M\) is an oriented Riemannian manifold with Riemannian connection \(\nabla\), then \(P_{\gamma(t), \gamma(t_0)}\) is an isometry preserving orientation.