TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Alexander Bobenko
Jan Techter

Complex Analysis II - WS 2015/2016

http://www3.math.tu-berlin.de/geometrie/Lehre/WS15/ComplexAnalysisII/

Exercise Sheet 11

Exercise 1: Elliptic function from Euclidean tiling. (6 pts)

Let Δ be the triangle with vertices $(b_1, b_2, b_3) = (1, i, 0)$.

(i) Determine the conformal map $f : \mathbb{H} \to \Delta$ that satisfies $f(b_1) = 0$, $f(b_2) = 1$, $f(b_3) = \infty$.

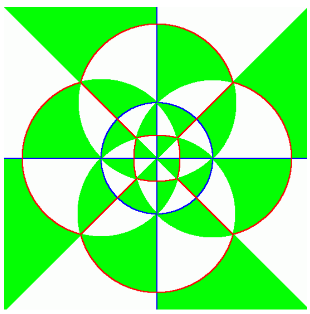
(ii) Consider the inverse map $g = f^{-1} : \Delta \to \mathbb{H}$ and use the Schwarz reflection principle to find its meromorphic continuation $g : \mathbb{C} \to \mathbb{H}$ if possible.
Determine all poles and zeros of g and their corresponding orders.

(iii) Show that g is elliptic and determine its periods.

(iv) Express g in terms of the Weierstraß $\wp$-function (up to a multiplicative constant). How can one determine this constant?

Exercise 2: Octahedral tiling. (6 pts)

Consider the octahedral tiling (where the blue circle has radius 1):



Now consider the meromorphic function g on $\mathbb{C}$ which maps the triangle of this tiling with vertices $b_3 = 0$, b_1 on the positive real axis and b_2 on the angle bisector of the first quadrant onto $\mathbb{H}$ and satisfies $g(0) = \infty$, $g(b_1) = 0$ and $g(b_2) = 1$.
Determine all points where g has zeros or poles together with their orders and show that

$$g(z) = A\frac{(z^8 - 34z^4 + 1)^2(z^4 + 1)^2}{z^4(z^4 - 1)^4}$$

(where $A = -1/108$ need not to be calculated).

Exercise 3: Linear differential equations with singularities. (8 pts)

(i) Locate and classify all singularities of the following differential equations:

(a) $zw'' + w' + zw = 0$

(b) $(2z + z^3)w'' - w' - 6zw = 0$.

Determine a series representation for a solution of (a) near the origin and determine its radius of convergence.

(ii) Consider a Fuchsian system

$$w' = \sum_{k=1}^{m} \frac{A_k}{z - a_k} w$$

with $A_k \in gl(n, \mathbb{C})$, $k = 1, ..., m$.

Transform the variable z by a Moebius transformation, that is $\zeta = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$. Show that the transformed system of differential equations is again Fuchsian.

Hint: It suffices to show this for $\zeta = az + b$ and $\zeta = \frac{1}{z}$. Why?

(iii) Let $a \in \mathbb{C}$. Show that $z_0 = \infty$ is a Fuchsian singularity of the differential equation

$$(1 - z^2)w'' - 2zw' + a(a + 1)w = 0.$$

Due Thursday, January 28, before the tutorial.