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http://www3.math.tu-berlin.de/geometrie/Lehre/WS15/DGII/ WS 15

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 4

(Vector fields, submanifolds)

due 16.11.2015

Exercise 1 **5 points**

On $\mathbb{S}^2 = \{x = (x_0, x_1, x_2) \mid \|x\| = 1\} \subset \mathbb{R}^3$ we consider coordinates given by the stereographic projection from the north pole $N = (1, 0, 0)$:

$$y_1 = \frac{x_1}{1 - x_0}, \qquad y_2 = \frac{x_2}{1 - x_0}.$$

Let the vector fields X and Y on $\mathbb{S}^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}, \qquad Y = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole $S = (-1, 0, 0)$.

Exercise 2 **5 points**

Show that the Möbius band (without boundary)

$$M = \left\{\left((2 + r\cos\tfrac{\varphi}{2})\cos\varphi, (2 + r\cos\tfrac{\varphi}{2})\sin\varphi, r\sin\tfrac{\varphi}{2}\right) \mid r \in (-\tfrac{1}{2}, \tfrac{1}{2}),\ \varphi \in \mathbb{R}\right\}$$

is a submanifold of $\mathbb{R}^3$. Show further that for each $p \in \mathbb{R}\mathrm{P}^2$ the open subset $\mathbb{R}\mathrm{P}^2 \setminus \{p\} \subset \mathbb{R}\mathrm{P}^2$ is diffeomorphic to M.

Exercise 3 **5 points**

Show that $\mathrm{SO}(n)$ is a smooth submanifold of the real $n \times n$ matrices $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. Compute its tangent space at the identity.