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http://www3.math.tu-berlin.de/geometrie/Lehre/WS15/DGII/ WS 15

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 7

(Riemannian manifolds)

due 7.12.2015

Definition (Isometry). Suppose (M,g) and $(N,\tilde{g})$ are Riemannian manifolds. A smooth map $f\colon M \to N$ is called an *isometry* if it is a diffeomorphism which satisfies

$$f^*\tilde{g} = g.$$

We call (M,g) and $(N,\tilde{g})$ *isometric* if there exists an isometry $f\colon M \to N$.

Exercise 1 **5 points**

Let $\mathrm{SU}(2) \subset \mathbb{C}^{2\times 2}$ denote the *special unitary group*, i.e. the group of unitary 2×2 matrices with determinant 1:

$$\mathrm{SU}(2) = \{ \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2} \mid |a|^2 + |b|^2 = 1\}.$$

a) Show that the scalar product $\langle X, Y\rangle := \frac{1}{2}\mathrm{trace}(\bar{X}^t Y)$ on $\mathrm{gl}(2,\mathbb{C})$ defines a Riemannian metric on $\mathrm{SU}(2)$.
b) Show that the left multiplication $L_A\colon B \mapsto AB$ and the right multiplication $R_A\colon B \mapsto BA$ by a constant $A \in \mathrm{SU}(2)$ are isometries of $\mathrm{SU}(2)$.
c) Show that $\mathrm{SU}(2)$ and the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ (with induced metric) are isometric.

Exercise 2 **5 points**

Let g_0 denote the Euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$. Let

- $M = \{z \in \mathbb{C} \mid |z| < 1\}$ be equipped with the Riemannian metric $g = \frac{4}{(1-|z|^2)^2} g_0$,
- $N = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ be equipped with the Riemannian metric $\tilde{g} = \frac{1}{(\mathrm{Im}\, z)^2} g_0$.

Show that the following map is an isometry:

$$f\colon M \to N, \quad z \mapsto i\frac{1-z}{1+z}.$$

Exercise 3 **5 points**

Let $f\colon M \to (N,g)$ be an immersion and let $\gamma\colon \mathbb{R} \supset I \to M$ be a smooth curve. Show that the length of γ with respect to the pull-back metric f^*g on M satisfies

$$\mathrm{len}_{f^*g}(\gamma) = \mathrm{len}_g(f\circ\gamma).$$