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http://www3.math.tu-berlin.de/geometrie/Lehre/WS15/DGII/ WS 15

# Differential Geometry II: Analysis and Geometry on Manifolds

## Exercise Sheet 8

(Forms, isometric immersion, tensor product)

due 14.12.2015

**Exercise 1** **5 points**

Is there a 1-form $\omega$ on $\mathbb{S}^2$ that vanishes at exactly one point? If so, can it chosen to be *exact*, i.e. $\omega = df$ for some function $f$?

**Exercise 2** **5 points**

Consider $\mathbb{S}^2 \subset \mathbb{R}^3$ with its induced metric. Show that the map

$$f\colon \mathbb{S}^2 \to \mathbb{R}^6, \quad (x, y, z) \mapsto (\tfrac{1}{\sqrt{2}}x^2, \tfrac{1}{\sqrt{2}}y^2, \tfrac{1}{\sqrt{2}}z^2, xy, xz, yz)$$

is an isometric immersion in the sense that the metric on $\mathbb{R}^6$ pulls back to the metric on $\mathbb{S}^2$. To what extend can $f$ be considered as isometric immersion from the real projective plane $\mathbb{RP}^2$ to $\mathbb{RP}^5$.

**Exercise 3** **5 points**

Let $V$ and $W$ be finite-dimensional real vector spaces and define

$$A := \big\{\beta\colon V^* \times W^* \to \mathbb{R} \mid \beta \textit{ bilinear}\big\}.$$

Clearly, $\otimes\colon V \times W \to A$, $(v \otimes w)(\omega, \eta) := \omega(v)\eta(w)$ is bilinear. Show:

a) If $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ are bases of $V$ and $W$ (resp.), then $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$ is a basis of $A$.

b) For each bilinear map $\mu\colon V \times W \to X$ into a real vector space $X$ there is a unique linear map $\lambda\colon A \to X$ such that $\mu = \lambda \circ \otimes$.

This shows that $A = V \otimes W$.