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http://www3.math.tu-berlin.de/geometrie/Lehre/WS15/DGII/ WS 15

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 9

(Forms, exterior derivative)

due 11.1.2016

Exercise 1 **5 points**

Show that the pullback commutes with the wedge product in the sense that

$$f^*(\eta \wedge \omega) = (f^*\eta) \wedge (f^*\omega)$$

for $f\colon M \to N$ and $\eta, \omega \in \Omega^*(N)$.

Exercise 2 **5 points**

Let $M = \mathbb{R}^3$. Determine which of the following forms are closed ($d\omega = 0$) and which are exact ($\omega = d\theta$ for some θ):

a) $\omega = yz\,dx + xz\,dy + xy\,dz$,

b) $\omega = x\,dx + x^2y^2\,dy + yz\,dz$,

c) $\omega = 2xy^2\,dx \wedge dy + z\,dy \wedge dz$.

If ω is exact, please write down the potential form θ explicitly.

Exercise 3 **5 points**

Consider $M = \mathbb{R}^n$ with the standard metric and standard orientation. To any vector field $\xi \in \Gamma(TM)$ associate the one-form $\omega_\xi \in \Omega^1(M)$ defined by

$$\omega_\xi(X) = \langle X, \xi \rangle$$

for $X \in T_pM = \mathbb{R}^n$. Let $\Omega \in \Omega^n(M)$ denote the Riemannian volume form and ι_ξ be the contraction with ξ.

a) Show that the Hodge star of ω_ξ is given by $\star\omega_\xi = \iota_\xi\Omega$.

b) Show that $\omega_{\operatorname{grad} f} = df$ for any $f \in \mathscr{C}^\infty(M)$.

c) Show that $d\big(\star\omega_\xi\big) = \operatorname{div}(\xi)\,\Omega$ for any vector field ξ.

d) For $n = 3$ show that $d\omega_\xi = \star\omega_{\operatorname{curl}\xi}$ for any vector field ξ.