

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 10

(Orientability, integration)

due 18.1.2016

Exercise 1

5 points

For which $n \in \mathbb{N}$ is $\mathbb{R}\mathbb{P}^n$ orientable, and for which not? Prove it.

Exercise 2

5 points

Prove that an n -dimensional manifold M for which there exists an immersion $f: M \rightarrow \mathbb{R}^{n+1}$ is orientable if and only if there is a smooth nowhere-vanishing normal vector field along (M, f) , i.e. a map $N: M \rightarrow \mathbb{S}^n$ such that $N(p) \perp d_p f(T_p M)$ for all $p \in M$.

Exercise 3

5 points

Consider \mathbb{R}^n with standard metric $\langle \cdot, \cdot \rangle$. Let $M \subset \mathbb{R}^n$ be an n -dimensional submanifold with boundary with outward pointing normal vector field $N: \partial M \rightarrow \mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then the volume form of M is given by $\omega_M = \det$ and the induced volume form on ∂M is given by $\omega_{\partial M} = \iota_N \omega_M$. Let $X \in \Gamma(T\mathbb{R}^n)$. Show that

$$\int_M \operatorname{div}(X) \omega_M = \int_{\partial M} \iota_X \omega_M = \int_{\partial M} \langle X, N \rangle \omega_{\partial M}.$$