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Geometry I - WS 15

http://www.math.tu-berlin.de/geometrie/Lehre/WS15/GeometryI/

Exercise Sheet 4

Exercise 1. (6 pts)

Consider the orthogonal group $O(1,1)$ acting on the Lorentz space $\mathbb{R}^{1,1}$ with its scalar product $\langle x, y\rangle := x_1y_1 - x_2y_2$.

- Show that any v with $\langle v, v\rangle = 1$ can be written as $v = (\pm\cosh t, \sinh t)$ for some $t \in \mathbb{R}$, while any v with $\langle v, v\rangle = -1$ can be written as $v = (\sinh t, \pm\cosh t)$.
- Consider the family of matrices
$$R_t := \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \in O(1,1).$$
Show that for any $s, t \in \mathbb{R}$ we have $R_sR_t = R_{s+t}$.
- Show that every matrix in $O(1,1)$ can be written as DR_t where D is one of the four possible diagonal matrices with diagonal entries ± 1 and R_t is as above for some $t \in \mathbb{R}$.

Exercise 2. (4 pts)

Let V be the four-dimensional vector space of real 2×2 matrices. The determinant defines a quadratic form $Q(M) := \det M$ on V. Find the associated symmetric bilinear form $B(M, N)$ on V (such that $Q(M) = B(M, M)$). Find a basis for V so that the matrix of B is in *standard form*: it is diagonal and the diagonal entries are elements of the set $\{1, -1, 0\}$, in that order. What is the signature of B?

Exercise 3. (4 pts)

Given a subset $U \subset \mathbb{R}^{n,1}$ in Lorentz space, let $U^\perp$ denote its orthogonal complement
$$U^\perp := \{x \in \mathbb{R}^{n,1} : \langle x, u\rangle = 0 \quad \forall u \in U\}.$$
(a) Show that $U^\perp$ is a vector subspace.
(b) Let $S_0 \subset \mathbb{R}^{n,1}$ denote the cone of light-like vectors. What is $S_0^\perp$?
(c) Prove that, $U \subset (U^\perp)^\perp$ for any $U \subset \mathbb{R}^{n,1}$.
(d) Does equality always hold in the above?

Exercise 4. (2 pts)

Prove that the columns of an orthogonal matrix in $O(p, q)$ form an orthonormal basis with respect to the scalar product on $\mathbb{R}^{p,q}$.

Due: before lecture on Tuesday, November 17, 2015.