TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I - WS 15

http://www.math.tu-berlin.de/geometrie/Lehre/WS15/GeometryI/

Exercise Sheet 9 and 10

Exercise 1: Involutions in $\mathbb{R}\mathrm{P}^1$. (3 pts)

Given a projective transformation f of $\mathbb{R}\mathrm{P}^1$, show that f can be factored as the product (composition) of at most three projective involutions. (An involution is a projective map $g \neq \mathrm{id}$ with $g^2 = g \circ g = \mathrm{id}$.)

Exercise 2: Projective transformations in $\mathbb{R}\mathrm{P}^1$. (3 pts)

Let $Q := \{[0,1],[1,0],[1,2],[2,1]\} \subset \mathbb{R}\mathrm{P}^1$. Consider the projective transformations $\tau : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ such that $\tau(Q) \subseteq Q$.

(a) How many such transformations τ exist?
(b) Find explicit formulas for these transformations τ.

Exercise 3: Projective transformations in $\mathbb{R}\mathrm{P}^2$. (4 pts)

Consider the following vectors in $\mathbb{R}^3$:

$$\begin{aligned} a_1 &= (1,0,1), \quad a_2 = (0,1,1), \quad a_3 = (0,0,1), \quad a_4 = (1,1,1), \\ b_1 &= (1,0,0), \quad b_2 = (0,1,0), \quad b_3 = a_3, \quad b_4 = a_4. \end{aligned}$$

Let $\hat{f}$ be the projective transformation of $\mathbb{R}\mathrm{P}^2$ such that $\hat{f}([a_i]) = [b_i]$ for $i \in \{1,2,3,4\}$. Find a linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ generating $\hat{f}$.

Exercise 4: Cross-ratio. (2 pts)

Find all $t \in \mathbb{C}$ such that there exists a projective transformation $\tau : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ with $\tau(0) = 0, \tau(1) = 1, \tau(t) = 2, \tau(2) = 6 - t$.

Exercise 5: Pappus. (4 pts)

Let $P_1, P_2, P_3, P_4, P_5, P_6$ be distinct points in the projective plane $\mathbb{R}\mathrm{P}^2$. Suppose that the three lines P_1P_2, P_4P_5, P_3P_6, as well as the three lines P_2P_3, P_5P_6, P_4P_1 intersect at one point. Show that the lines P_3P_4, P_6P_1, P_5P_2 also intersect at one point.
Tip: Find Pappus.

Exercise 6: Desargues' theorem in 3D. (4 pts)

Let P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 be points in general position in $\mathbb{R}\mathrm{P}^3$, forming two tetrahedra. Suppose that the four lines P_iQ_i connecting corresponding vertices intersect in a common point. Given a pair of corresponding faces, the planes they lie in intersect in a line. Show that all four such lines lie in some common plane.

Additional Christmas Exercise! (2 bonus pts)

Simplify the equation such that the text makes sense.

$$\text{WE WISH YOU A } y = \frac{\ln(\frac{x}{m} - as)}{r^2} \text{ AND A HAPPY NEW YEAR!}$$

Due: Before lecture on Tuesday 12 January 2016.