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http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/DGII/ WS 17

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 4

(Vector fields, tangent bundle)

due 20.11.2017

Exercise 1 **5 points**

Show that each of the following conditions is equivalent to the smoothness of a vector field X as a section $X\colon M \to \mathrm{T}M$:

a) For each $f \in \mathscr{C}^\infty(M)$, the function Xf is also smooth.
b) If we write $X|_U =: \sum v_i \frac{\partial}{\partial x_i}$ in a coordinate chart $\varphi = (x_1, \ldots, x_n)$ defined on $U \subset M$, then the components $v_i\colon U \to \mathbb{R}$ are smooth.

Exercise 2 **5 points**

On $\mathbb{S}^2 = \{x = (x_0, x_1, x_2) \mid \|x\| = 1\} \subset \mathbb{R}^3$ we consider coordinates given by the stereographic projection from the north pole $N = (1, 0, 0)$:

$$y_1 = \tfrac{x_1}{1-x_0}, \quad y_2 = \tfrac{x_2}{1-x_0}.$$

Let the vector fields X and Y on $\mathbb{S}^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2 \tfrac{\partial}{\partial y_1} - y_1 \tfrac{\partial}{\partial y_2}, \qquad Y = y_1 \tfrac{\partial}{\partial y_1} + y_2 \tfrac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole $S = (-1, 0, 0)$.

Exercise 3 **5 points**

Prove that the tangent bundle of a product of smooth manifolds is diffeomorphic to the product of the tangent bundles of the manifolds. Deduce that the tangent bundle of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$.