Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 5

(Lie bracket, vector bundles, connections)

due 27.11.2017

Exercise 1 **5 points**

Calculate the commutator $[X, Y]$ of the following vector fields on $\mathbb{R}^2 \setminus \{0\}$:

$$X = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}, \qquad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Write X and Y in polar coordinates $(r\cos\varphi, r\sin\varphi) \mapsto (r, \varphi)$.

Exercise 2 **5 points**

Let $M \subset \mathbb{R}^k$ be a smooth submanifold of dimension n. Let $\iota\colon M \hookrightarrow \mathbb{R}^k$ denote the inclusion map. Show that the normal bundle $\mathrm{N}M = \dot\bigcup_{p\in M}(\mathrm{T}_pM)^\perp \subset \iota^*\mathrm{T}\mathbb{R}^k \cong M \times \mathbb{R}^k$ is a smooth rank $k-n$ vector bundle over M.

Exercise 3 **5 points**

Let V_i and W denote vector bundles with connections ∇^i and ∇, respectively. Show that the equation

$$(\hat\nabla_X T)(Y_1,\dots,Y_n) = \nabla_X(T(Y_1,\dots,Y_n)) - \sum_i T(Y_1,\dots,\nabla^i_X Y_i,\dots,Y_n)$$

for $T \in \Gamma(V_1^* \otimes \cdots \otimes V_n^* \otimes W)$ and vector fields $Y_i \in \Gamma(V_i)$ defines a connection $\hat\nabla$ on the bundle of multilinear forms $V_1^* \otimes \cdots \otimes V_n^* \otimes W$.