Technische Universität Berlin
Institut für Mathematik
Pinkall / Knöppel
http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/DGII/ WS 17

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 6

(Vector bundles, connections, differential forms)

due 4.12.2017

Exercise 1 **5 points**

Show that the tangent bundle $\mathrm{T}\mathbb{S}^3$ of the round sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ is trivial.

Hint: Show that the vector fields $\varphi_1(x_1, x_2, x_3, x_4) = (-x_2, x_1, x_4, -x_3)$, $\varphi_2(x_1, x_2, x_3, x_4) = (x_3, x_4, -x_1, -x_2)$ and $\varphi_3(x_1, x_2, x_3, x_4) = (-x_4, x_3, -x_2, x_1)$ form a frame of $\mathrm{T}\mathbb{S}^3$.

Exercise 2 **5 points**

Let ∇ be a connection on a direct sum $E = E_1 \oplus E_2$ of two vector bundles over M. Show that

$$\nabla = \begin{pmatrix} \nabla^1 & A \\ \tilde{A} & \nabla^2 \end{pmatrix},$$

where $\tilde{A} \in \Omega^1(M, \mathrm{Hom}(E_1, E_2))$, $A \in \Omega^1(M, \mathrm{Hom}(E_2, E_1))$ and ∇^i are connections on the bundles E_i.

Exercise 3 **5 points**

Let $M = \mathbb{R}^2$. Let $J \in \Gamma(\mathrm{End}\mathrm{T}M)$ be the 90° rotation and $\det \in \Omega^2(M)$ denote the determinant. Define $*\colon \Omega^1(M) \to \Omega^1(M)$ by $*\omega(X) = -\omega(JX)$. Show that

a) for all $f \in \mathscr{C}^\infty(M)$, $d * df = (\Delta f) \det$, where $\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f$,

b) $\omega \in \Omega^1(M)$ is *closed* (i.e. $d\omega = 0$), if and only if ω is *exact* (i.e. $\omega = df$).