Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 7

(Connections and differential forms)

due 11.12.2017

Exercise 1 **5 points**

Let $M \subset \mathbb{R}^2$ be open. On $E = M \times \mathbb{R}^2$ we define two connections ∇ and $\tilde{\nabla}$ as follows:

$$\nabla = d + \begin{pmatrix} 0 & -x\,dy \\ x\,dy & 0 \end{pmatrix}, \quad \tilde{\nabla} = d + \begin{pmatrix} 0 & -x\,dx \\ x\,dx & 0 \end{pmatrix}.$$

Show that (E, ∇) is not trivial. Further construct an explicit isomorphism between $(E, \tilde{\nabla})$ and the trivial bundle (E, d).

Exercise 2 **5 points**

Let $M = \mathbb{R}^3$. Determine which of the following forms are closed ($d\omega = 0$) and which are exact ($\omega = d\theta$ for some θ):

a) $\omega = yz\,dx + xz\,dy + xy\,dz$,
b) $\omega = x\,dx + x^2y^2\,dy + yz\,dz$,
c) $\omega = 2xy^2\,dx \wedge dy + z\,dy \wedge dz$.

If ω is exact, please write down the potential form θ explicitly.

Exercise 3 **5 points**

Let $M = \mathbb{R}^n$. For $\xi \in \Gamma(TM)$, we define $\omega^\xi \in \Omega^1(M)$ and $*\omega^\xi \in \Omega^{n-1}(M)$ as follows:

$$\omega^\xi(X_1) := \langle \xi, X_1 \rangle, \quad *\omega^\xi(X_2, \dots, X_n) := \det(\xi, X_2, \dots, X_n), \quad X_1, \dots, X_n \in \Gamma(\mathrm{T}M).$$

Show the following identities:

$$df = \omega^{\operatorname{grad} f}, \quad d * \omega^\xi = \operatorname{div}(\xi) \det,$$

and for $n = 3$,

$$d\omega^\xi = *\omega^{\operatorname{rot}\xi}.$$