Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 8

(Parallel transport, Riemannian manifolds)

due 18.12.2017

Exercise 1 **5 points**

Let $M \subset \mathbb{R}$ be an interval and consider the vector bundle $E = M \times \mathbb{R}^k$, $k \in \mathbb{N}$, equipped with some connection ∇. Show that (E, ∇) is trivial.

Exercise 2 **5 points**

Let (M, g) be a Riemannian manifold and $\tilde{g} = e^{2u}g$ for some smooth function $u\colon M \to \mathbb{R}$. Show that between the corresponding Levi-Civita connections the following relation holds:

$$\tilde{\nabla}_X Y = \nabla_X Y + du(X)Y + du(Y)X - g(X,Y)\operatorname{grad} u.$$

Here $\operatorname{grad} u \in \Gamma(TM)$ is the vector field uniquely determined by the condition $du(X) = g(\operatorname{grad} u, X)$ for all $X \in \Gamma(TM)$.

Exercise 3 **5 points**

Let $(M, \langle .,. \rangle)$ be a 2-dimensional Riemannian manifold, ∇ its Levi-Civita connection. Show that there is a function $K \in \mathscr{C}^\infty(M)$ such that

$$R^\nabla(X,Y)Z = K\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big), \; \textit{for all } X, Y, Z \in \Gamma(\mathrm{T}M).$$