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http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/DGII/ WS 17

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 13

(Conjugate Points, Hadamard-Cartan theorem)

due 05.02.2018

Definition

Let (M, g) be a Riemannian manifold and $\gamma\colon [a, b] \to M$ be a geodesic with end points p and q. The points p and q are said to be *conjugate along* γ if there exists a non-trivial Jacobi field Y along γ such that $Y(a) = 0 = Y(b)$.

Exercise 1 — 8 points

Let $\gamma\colon [a, b] \to M$ be a geodesic in a Riemannian manifold (M, g). If there exists $s_0 \in (a, b)$ such that $\gamma(a)$ and $\gamma(s_0)$ are conjugate along γ, then there is a variation γ_t of γ with fixed end points such that for t small enough

$$L(\gamma_t) < L(\gamma) \textit{ and } E(\gamma_t) < E(\gamma).$$

Hint: Construct a (piecewise smooth) vector field along γ such that the second variation is negative.

Exercise 2 — 7 points

Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature. Show that for every $m \in M$ there is a metric $\tilde{g}$ on T_mM such that $\exp_m$ is an isometric immersion and $(T_mM, \tilde{g})$ is complete.

Remark: Since an isometric immersion from a complete Riemannian manifold into a Riemannian manifold of equal dimension is always a covering map, we obtain that in the case of non-positive sectional curvature the map $\exp_m$ is a covering map for each $m \in M$. This is the Hadamard-Cartan theorem.