TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I - WS 2017/2018

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/GeometryI/

Exercise Sheet 7

Exercise 1: Degenerate quadrics. (4 pts)

Let q be a degenerate, but non-vanishing quadratic form on $\mathbb{R}^{n+1}$, which defines the quadric $Q \subset \mathbb{RP}^n$. Let b be the corresponding symmetric bilinear form and denote $U_0 = \ker q = \{u \in \mathbb{R}^{n+1} \mid b(u, v) = 0 \ \forall v \in \mathbb{R}^{n+1}\}$. Consider any complementary subspace U_1 of U_0, such that $\mathbb{R}^{n+1} = U_0 \oplus U_1$.
Denote $Q_1 \subset P(U_1)$ the non-degenerate quadric Q_1 that is defined by the restriction $q|_{U_1}$. Under the assumption $Q_1 \neq \emptyset$, show that Q is the union of all lines joining any point in $P(U_0)$ with any point on Q_1. Draw a sketch that illustrates this decomposition in $\mathbb{RP}^3$. What happens, if Q_1 is empty?

Exercise 2: Polarity of lines in $\mathbb{RP}^3$. (4 pts)

Let $\mathcal{Q}$ be a non-degenerate quadric in $\mathbb{RP}^3$, $\ell \subset \mathbb{RP}^3$ a line and $\ell^\perp \subset \mathbb{RP}^3$ its polar. Show that all polar planes of points on ℓ pass through $\ell^\perp$.

Exercise 3: Sphere polarity. (4 pts)

For fixed affine coordinates of $\mathbb{RP}^3 = \mathbb{R}^3 \cup \mathbb{RP}^2$, consider the quadric Q in $\mathbb{RP}^3$ whose affine image is the unit sphere $S^2 \subset \mathbb{R}^3$. Polarity with respect to Q gives not only a relation between points and planes in $\mathbb{RP}^3$, but also a relation between lines: for any line l one obtains the polar line $l^\perp$.
What is the signature of the quadric Q? Find a way to construct the line $l^\perp$ from a given line l. Describe and prove the geometric relation between l and $l^\perp$ in $\mathbb{R}^3$ and draw a sketch.

Exercise 4: Brianchon and Pascal in $\mathbb{RP}^3$. (4 pts)

A non-planar hexagon $ABCDEF$ in $\mathbb{RP}^3$ is called a *Brianchon hexagon* if its diagonals AD, BE, CF meet in one point. It is called a *Pascal hexagon* if the intersection lines $ABC \cap DEF$, $BCD \cap EFA$, and $CDE \cap FAB$ lie in one plane.
Show that a non-planar hexagon is a Brianchon hexagon if and only if it is a Pascal hexagon. What is the relation between the Brianchon point and the Pascal plane with respect to the quadric that contains the hexagon?

Hint: Show that a non-planar hexagon is a Brianchon hexagon if and only if all its sides lie on a quadric.

Due Wednesday, December 13.