Technische Universität Berlin
Institut für Mathematik

Mathematical Physics I, Dr. Matteo Petrera, Mats Vermeeren WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

Exercise Sheet 10 (18.01.18)

Due date: ***25.01.18***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students.*
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (2+2+2 pts)

Prove that the following transformations are canonical by showing that they preserve the symplectic 2-form.

1. $f_1 : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \sqrt{p} \\ -2q\sqrt{p} \end{pmatrix}, \qquad \text{for } p > 0.$

2. $f_2 : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \qquad \varphi \in [0, 2\pi).$

3. $f_3 : \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \mapsto \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}.$

Exercise 2 (4+2+4 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^4$ consider the discrete dynamical system defined by iterations of the map $\Phi : \mathbb{R}^4 \setminus \{q_1^2 = q_2^2\} \to \mathbb{R}^4$ given by:

$$\widetilde{q}_1 = p_1(q_1^2 + q_2^2) + 2\, q_1\, q_2\, p_2, \qquad \widetilde{p}_1 = \frac{q_1}{q_1^2 - q_2^2},$$
$$\widetilde{q}_2 = p_2(q_1^2 + q_2^2) + 2\, q_1\, q_2\, p_1, \qquad \widetilde{p}_2 = -\frac{q_2}{q_1^2 - q_2^2}.$$

It can be proved that the map Φ preserves the canonical Poisson brackets.

1. Prove that the functions

$$F_1(q_1, q_2, p_1, p_2) := q_1\, p_1 + q_2\, p_2,$$
$$F_2(q_1, q_2, p_1, p_2) := q_1\, p_2 + q_2\, p_1,$$

are two functionally independent integrals of motion of Φ.

Turn over

2. Prove that F_1 and F_2 are in Poisson involution, i.e., $\{ F_1, F_2 \} = 0$

Consider the change of coordinates $\Psi : (q_1, q_2, p_1, p_2) \mapsto (Q_1, Q_2, P_1, P_2)$ defined by:

$$Q_1 = \frac{1}{2}\big(\log(q_1 + q_2) + \log(q_1 - q_2)\big), \qquad P_1 = q_1\, p_1 + q_2\, p_2,$$
$$Q_2 = \frac{1}{2}\big(\log(q_1 + q_2) - \log(q_1 - q_2)\big), \qquad P_2 = q_1\, p_2 + q_2\, p_1.$$

It can be proved that the map Ψ preserves the canonical Poisson brackets.

3. Prove that Ψ allows to *linearize* the map Φ:

$$\widetilde{Q}_1 = Q_1 + \nu_1(F_1, F_2), \qquad \widetilde{P}_1 = P_1,$$
$$\widetilde{Q}_2 = Q_2 + \nu_2(F_1, F_2), \qquad \widetilde{P}_2 = P_2.$$

Here ν_1 and ν_2 are two functions of the integrals of motion to be determined.

Exercise 3 (1+2+3+3 pts)

In Ex. 3 of Sheet 8, we encountered a Lagrangian of the form

$$\mathcal{L}(q, \dot{q}) = -\sqrt{1 - \langle \dot{q}, \dot{q} \rangle} - \langle a, q \rangle,$$

where $(q, \dot{q}) \in \mathbb{R}^6$. The Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{q}}{\sqrt{1 - \langle \dot{q}, \dot{q} \rangle}}\right) + a = 0, \tag{1}$$

and the corresponding canonical Hamilton equations are

$$\dot{q} = \frac{p}{\sqrt{1 + \langle p, p \rangle}}, \qquad \dot{p} = -a. \tag{2}$$

Now consider the following discretization of the Lagrangian $\mathcal{L}$:

$$\mathbf{L}\big(q, \widetilde{q}\,\big) = \epsilon \left(-\sqrt{1 - \frac{\langle \widetilde{q} - q, \widetilde{q} - q \rangle}{\epsilon^2}} - \frac{1}{2}\langle a, q + \widetilde{q}\rangle\right), \qquad 0 < \epsilon \ll 1,$$

where $q = q(n)$, $\widetilde{q} = q(n+1)$, $n \in \mathbb{N}$. Note that first derivative $\dot{q}$ is discretized by the finite difference $(\widetilde{q} - q)/\epsilon$. Similarly for $\dot{p}$.

1. Show that $\mathbf{L}\big(q, \widetilde{q}\,\big) = \epsilon\, \mathcal{L}(q, \dot{q}) + O(\epsilon^2)$.
2. Compute the explicit form of the symplectic map defined by the equations

$$p = -\operatorname{grad}_q \mathbf{L}\big(q, \widetilde{q}\,\big), \qquad \widetilde{p} = \operatorname{grad}_{\widetilde{q}} \mathbf{L}\big(q, \widetilde{q}\,\big).$$

3. Find the discrete analogue of equations (2). In other words, compute the quantities $(\widetilde{q} - q)/\epsilon$ and $(\widetilde{p} - p)/\epsilon$. Check that they converge to (2) in the continuous limit.
4. Construct the discrete Euler-Lagrange equations

$$\operatorname{grad}_q \mathbf{L}\big(q, \widetilde{q}\,\big) + \operatorname{grad}_q \mathbf{L}\big(\underset{\sim}{q}, q\big) = 0,$$

where $\underset{\sim}{q} = q(n-1)$. Check that they converge to (1) in the continuous limit.