Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, Mats Vermeeren WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

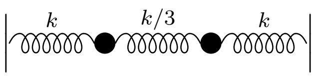
Exercise Sheet 2 (26.10.17)

Due date: ***09.11.17***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students.*
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (2+4+1 pts)

Consider a one-dimensional system consisting of two masses m and three springs with elasticity constants k, $k/3$ and k, $k > 0$. The springs with elasticity constant k are firmly attached to the wall, while the spring with elasticity constant $k/3$ connects the two masses.



x_1 x_2

Denote by x_1 and x_2 the positions of the masses. Their equilibrium positions are $x_1 = e_1$ and $x_2 = e_2$. Consider the initial conditions

$$(x_1(0), x_2(0)) = (a, b), \qquad (\dot{x}_1(0), \dot{x}_2(0)) = (c, d),$$

with $b > a > 0$ and $c, d > 0$.

1. According to Hooke law, express the spring forces acting on each mass (frictional forces are neglected) and write down the coupled system of Newton equations of motion for the two masses.

   (Hint: Hooke law states that the spring force is $f(x) = -k\,x$, where $k > 0$ is the elasticity constant and x is the displacement of the end of the spring from its equilibrium position)

2. Decouple the obtained differential equations by introducing new variables $(\widetilde{x}_1, \widetilde{x}_2) := (x_1 + x_2, x_1 - x_2)$. Solve the IVP in the variables $\widetilde{x}_1$, $\widetilde{x}_2$.

3. Determine explicitly the flow of the system in terms of the variables x_1, x_2.

Turn over

Exercise 2 (5 pts)

Consider the following IVP in $\mathbb{R}$:

$$\begin{cases} \dot{x} = \dfrac{e^t\left(1 - t\,x^2\right) - 1}{x\,t^2\,e^t}, \\ x(1) = 2. \end{cases}$$

Find the solution.

(Hint: After an appropriate change of coordinates the ODE is solvable by separation of variables.)

Exercise 3 (2+2+2 pts)

Consider the map $\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\Phi_t(x, y, z) \mapsto \left(x + 2ty,\ y,\ e^{-tx - t^2 y} z\right).$$

1. Prove that Φ_t is the flow of a continuous dynamical system.
2. Compute the infinitesimal generator of Φ_t.
3. Write down the IVP corresponding to Φ_t.

Exercise 4 (1+2 pts)

Fix $\alpha > 0$ and consider a discrete dynamical system defined by iterations of the map

$$\Phi : \left\{(x, y) \in \mathbb{R}^2 \ :\ x \neq \frac{1}{\alpha}\right\} \to \mathbb{R}^2 : (x, y) \mapsto \left(\frac{x}{\alpha\,x - 1}, \frac{y + \alpha\,x(x - y)}{\alpha\,x - 1}\right).$$

1. Show that the image of Φ is contained in $\left\{(x, y) \in \mathbb{R}^2 \ :\ x \neq \frac{1}{\alpha}\right\}$, so the iteration can be continued indefinitely.
2. Show that each orbit is periodic and compute the period.

Exercise 5 (4 pts)

Let $(\Phi^t)_{t\in\mathbb{R}}$, with $\Phi^t : M \to M \subset \mathbb{R}^n$, be a continuous dynamical system. Consider a point $x_0 \in M$ and consider the orbit $\mathcal{O}(x_0) := \{\Phi^t(x_0) : t \in \mathbb{R}\} \subset M$. Assume that $\Phi^T(x_0) = x_0$ and that T is the smallest positive number with this property. Prove that $\mathcal{O}(x_0)$ is a periodic orbit with period T.