Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, Mats Vermeeren WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

Exercise Sheet 3 (09.11.17)

Due date: ***16.11.17***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students*.
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (2+2+2 pts)

Consider the following two one-parameter maps acting on $\mathbb{R}^2$ ($t \in \mathbb{R}$):

$$\Phi_t : (x_1, x_2) \mapsto \frac{(x_1 - (x_1^2 + x_2^2)t, x_2)}{(1 - tx_1)^2 + t^2 x_2^2} \qquad \text{and} \qquad \Psi_t : (x_1, x_2) \mapsto \left(x_1 + t, \frac{x_1 x_2}{x_1 + t} \right).$$

1. Prove that both families Φ_t and Ψ_t consist of (local) diffeomorphisms which are (local) actions of $\mathbb{R}$ on $\mathbb{R}^2$.
2. Compute the vector fields generating the flows Φ_t and Ψ_t and write the associated ODEs.
3. Compute the Lie bracket of the vector fields from 2. Do the corresponding flows commute?

Exercise 2 (2+3 pts)

1. Consider the following IVP in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2), \\ (x_1(0), x_2(0)) \in \mathbb{R}^2, \end{cases}$$

   where $f_1, f_2 \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Note that the ODE

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \tag{1}$$

   can be formally obtained from the original system by dividing the second ODE by the first and then canceling the time differential dt. Let $F(x_1, x_2) = c$, $c \in \mathbb{R}$, be the solution of (1) in implicit form. Prove that F is an integral of motion of the IVP.

2. Use the procedure described in 1. to construct an integral of motion in the case

$$f_1(x_1, x_2) := 2\, x_1 - x_1 x_2, \qquad f_2(x_1, x_2) := -9\, x_2 + 3\, x_1 x_2.$$

   Sketch the phase portrait in the first quadrant.

TURN OVER!

Exercise 3 (5 pts)

Consider the dynamical system defined by the planar system of difference equations

$$\begin{cases} x_{k+1} - x_k = \alpha(x_{k+1}y_k + x_k y_{k+1}), \\ y_{k+1} - y_k = -2x_k x_{k+1}, \end{cases} \tag{2}$$

with $\alpha > 0$. The above system defines a map $(x_k, y_k) \mapsto (x_{k+1}, y_{k+1})$. Without finding the explicit form of the map, prove that the relation

$$\frac{x_k^2 + \alpha y_k^2}{1 + \alpha x_k^2} = \frac{x_{k+1}^2 + \alpha y_{k+1}^2}{1 + \alpha x_{k+1}^2}$$

holds on solutions of (2). In other words, the function

$$F(x_k, y_k) := \frac{x_k^2 + \alpha y_k^2}{1 + \alpha x_k^2}$$

is an integral of motion of the discrete dynamical system.

Exercise 4 (4 pts)

Consider the following IVP in $\mathbb{R}^3$:

$$\begin{cases} \dot{x}_1 = -x_1 x_3, \\ \dot{x}_2 = x_2 x_3, \\ \dot{x}_3 = x_1^2 - x_2^2, \\ (x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^3. \end{cases}$$

Find two functionally independent polynomial integrals of motion of the system.

Exercise 5 (2+3 pts)

Consider a Hamiltonian system of ODEs

$$\dot{x}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial x_k}, \qquad k = 1, \ldots, n \tag{3}$$

with some smooth function $H(x,p) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

1. Show that the phase space volume is preserved.
2. With the help of the Poincaré recurrence theorem, prove that (3) does not have asymptotically stable fixed points.