Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, Mats Vermeeren WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

Exercise Sheet 4 (16.11.17)

Due date: ***23.11.17***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students*.
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (2+1+2 pts)

Consider the discrete dynamical system defined in terms of iterations of the map $\Phi : \mathbb{R} \setminus \{1/2\} \to \mathbb{R}$ defined by

$$\Phi : x \mapsto \frac{3\,x-2}{2\,x-1}.$$

1. Construct the n-th iteration of Φ.
2. Show that $\lim_{n\to\infty} \Phi^n(x) = 1$, i.e., $x = 1$ is an attracting fixed point.
3. Is the point $x = 1$ asymptotically stable?

Exercise 2 (2+2 pts)

Consider the following ODE in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = -x_1^3 - 2x_1x_2^2, \\ \dot{x}_2 = x_1^2x_2 - x_2^3. \end{cases}$$

Consider the fixed point $(0, 0)$.

1. Is the function
$$F(x_1, x_2) := x_1^2 + x_1^2x_2^2$$
a Lyapunov function? What does it tell you about the stability of $(0, 0)$?
2. Is the function
$$G(x_1, x_2) := x_1^2 + x_1^2x_2^2 + x_2^4$$
a Lyapunov function? What does it tell you about the stability of $(0, 0)$?

TURN OVER!

Exercise 3 (3+2+2 pts)

Consider the following ODE in $\mathbb{R}^3$:

$$\begin{cases} \dot{x}_1 = 3\,x_2(x_3 - 1), \\ \dot{x}_2 = -x_1(x_3 - 1), \\ \dot{x}_3 = -x_3^3\left(x_1^2 + 1\right). \end{cases}$$

1. Linearize the system around the fixed point $(0,0,0)$ and determine the linear stability of the fixed point $(0,0,0)$. What does the Poincaré-Lyapunov theorem say about the stability of $(0,0,0)$ in the nonlinear system?
2. Prove that the fixed point $(0,0,0)$ is stable by finding a (quadratic) Lyapunov function.
3. Prove that the fixed point $(0,0,0)$ cannot be asymptotically stable by restricting the dynamics on the invariant plane $x_3 = 0$.

Exercise 4 (2+2+1 pts)

Consider the following IVP in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = Ax + \varepsilon x\|x\|^2, \\ x(0) \in \mathbb{R}^2 \end{cases}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \varepsilon \in \mathbb{R}.$$

Study the stability of the fixed point $(0,0)$ if

1. $\epsilon = 0$,
2. $\epsilon < 0$,
3. $\epsilon > 0$.

Exercise 5 (2+2 pts)

Consider the following planar linear systems of ODEs:

$$\begin{cases} \dot{x}_1 = -2x_1, \\ \dot{x}_2 = -3x_2, \end{cases} \qquad \begin{cases} \dot{x}_1 = -2x_1, \\ \dot{x}_2 = -2x_2. \end{cases}$$

1. Sketch the phase portraits of both systems.
2. Prove that the systems are topologically conjugate by constructing explicitly the map between the orbits.