Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, Mats Vermeeren WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

Exercise Sheet 5 (23.11.17)

Due date: ***30.11.17***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students*.
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (1+2 pts)

Consider the following system of ODEs in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = x_1\left(1 - x_1^2 - x_2^2\right) - x_2\left(1 + x_1^2 + x_2^2\right), \\ \dot{x}_2 = x_1\left(1 + x_1^2 + x_2^2\right) + x_2\left(1 - x_1^2 - x_2^2\right). \end{cases}$$

1. Rewrite the system of ODEs using polar coordinates, $(x_1, x_2) := r(\cos\theta, \sin\theta)$, $r > 0$, $\theta \in [0, 2\pi)$.
2. Find a periodic solution of the original system.

Exercise 2 (1+3+2+1 pts)

Consider the following system of ODEs in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = 2x_2 e^{-x_1^2 - x_2^2}, \\ \dot{x}_2 = \left(3x_1^2 - 6\right) e^{-x_1^2 - x_2^2}. \end{cases}$$

1. Find the fixed points.
2. Linearize the system around the fixed points.

   Only on the basis of the Poincaré-Lyapunov Theorem, what can you say about the stability of the fixed points for the nonlinear system?
3. Find an integral of motion $F(x_1, x_2)$. Use this to complete your study of the stability of the fixed points.

   Hint: use the Ansatz $F(x_1, x_2) = F_1(x_1) + F_2(x_2)$.
4. Sketch the phase portrait.

TURN OVER!

Exercise 3 (2+3+2 pts)

Consider the following system of ODEs in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = x_1^4 + x_1 x_2, \\ \dot{x}_2 = -2\,x_2 - x_1^2 + x_1 x_2^2. \end{cases} \tag{1}$$

1. Linearize (1) around the fixed point $(0,0)$. What can you say about the stability of $(0,0)$ on the basis of the Poincaré-Lyapunov Theorem?

2. Find the center (linear) space $E^c(0,0)$. Construct an approximation of the center manifold $W^c(0,0)$.

   Hint: The center manifold is parametrized, in a neighborhood of $(0,0)$, by $x_2 = h(x_1)$ for some function h. An approximation of $W^c(0,0)$ is given by the series expansion – say up to $O(x_1^5)$ – of the function h around $x_1 = 0$.

3. Find the first three nonzero terms of the series expansion of the system obtained by reducing (1) on $W^c(0,0)$. What can you say now about the stability of the fixed point $(0,0)$ for system (1)?

Exercise 4 (1+3+2+2 pts)

Consider the following system of ODEs in $\mathbb{R}^3$:

$$\begin{cases} \dot{x}_1 = \frac{1}{2}(x_3 - x_1) - x_2 + 2x_2^3, \\ \dot{x}_2 = \frac{1}{2}(x_3 - x_1), \\ \dot{x}_3 = x_3 - x_2^2(x_1 + x_3). \end{cases} \tag{2}$$

1. Find all fixed points.

2. Linearize system (2) around the fixed point $(0,0,0)$ and discuss the stability of $(0,0,0)$. Write the general solution of the linearized system.

3. Prove that $M := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 2x_2 + x_3 = 0\}$ is an invariant manifold for (2).

4. Reduce system (2) on M by eliminating the variable x_1, thus getting a first-order ODE for x_2 and x_3. Then eliminate the variable x_3, thus obtaining a second-order ODE for x_2. Find an integral of motion of the resulting ODE.

   Is this an integral of motion of the full system (2)?