Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, Mats Vermeeren WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

Exercise Sheet 6 (30.11.17)

Due date: ***7.12.17***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students*.
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (3+4 pts)

Consider the differential equation

$$\ddot{x} + \frac{a}{t}\dot{x} + \frac{b}{t^2}x = 0, \qquad (t > 0), \qquad a, b \in \mathbb{C}. \tag{1}$$

1. Prove that a function $\phi : \mathbb{R}_{>0} \to \mathbb{C}$ is a solution of (1) if and only if the function $\psi : \mathbb{R} \to \mathbb{C}$ defined by $\psi(\tau) := \phi(e^\tau)$ is a solution of the differential equation
$$\ddot{x} + (a-1)\dot{x} + bx = 0.$$
2. Determine a fundamental system of solutions of (1) for all possible values of $a, b \in \mathbb{C}$, i.e. two solutions ϕ_1, ϕ_2, such that
$$\begin{pmatrix} \phi_1 & \phi_2 \\ \dot{\phi}_1 & \dot{\phi}_2 \end{pmatrix}$$
is a fundamental matrix of the first order system corresponding to (1).

Exercise 2 (3 pts)

Consider the differential equation

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \ldots + a_1(t)\dot{x} + a_0(t)x = 0,$$

where $a_0, \ldots, a_{n-1} : J \to \mathbb{R}$ are continuous functions on an intervall $J \subset \mathbb{R}$. For n arbitrary solutions $x_1, \ldots, x_n$ we set

$$W(t) := W(x_1(t), \ldots, x_n(t)) := \begin{vmatrix} x_1(t) & \cdots & x_n(t) \\ \dot{x}_1(t) & \cdots & \dot{x}_n(t) \\ \vdots & & \vdots \\ x_1^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{vmatrix} \qquad (t \in J).$$

Prove that

$$W(t) = W(t_0) \exp\left(-\int_{t_0}^{t} a_{n-1}(\tau)d\tau\right) \qquad (t_0 \in J \text{ arbitrary}).$$

TURN OVER!

Exercise 3 (3+2+2 pts)

Consider the differential equation

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0, \tag{2}$$

where $a, b : \mathbb{R} \to \mathbb{R}$ are continuous and T-periodic functions. Let x_1, x_2 be a fundamental system of solutions of (2).

1. Prove that $W(t+T) = W(t) \det C$, where

   $$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

   is the monodromy matrix of (2) and W is defined as in Exercise 2.

2. From now on set $a(t) \equiv 0$. Prove that for the multipliers μ_1 and μ_2 there hold:

   $$\mu_1\mu_2 = 1 \qquad \text{and} \qquad \mu_1 + \mu_2 = c_{11} + c_{22}.$$

3. Consider a fundamental system x_1, x_2 of solutions of (2) $(a(t) \equiv 0)$ with

   $$x_1(0) = 1, \dot{x}(0) = 0 \qquad \text{and} \qquad x_2(0) = 0, \dot{x}_2(0) = 1.$$

   Prove that there exists an $r \in \mathbb{C}$ such that

   $$\mu_1 = e^r, \qquad \mu_2 = e^{-r}, \qquad \text{and} \qquad \cosh(r) = \frac{x_1(T) + \dot{x}_2(T)}{2}.$$

Exercise 4 (4+4 pts)

Consider the following two scalar ODEs:

1. $\dot{x} = x - \alpha\, x(1-x)$,
2. $\dot{x} = x + \dfrac{\alpha\, x}{1+x^2}$.

For both of them:

- Find and classify the bifurcations that occur as α is varied.
- Draw the bifurcation diagram.