Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

**Mathematical Physics I**, Dr. Matteo Petrera, Mats Vermeeren WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

# Exercise Sheet 7 (7.12.17)

*Due date:* ***14.12.17***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students.*
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

## Exercise 1 (3+3+4 pts)

1. Fix $T, \alpha > 0$. Consider the functional $\psi : K \to \mathbb{R}$ defined by
$$\psi(\gamma) := \int_0^T \dot{q}^2 \,\mathrm{d}t,$$
where $K$ is the space of all $C^1$-curves
$$\gamma = \left\{(t, q) \,:\, q = q(t),\, q \in C^1([0,T], \mathbb{R}),\, q(0) = 0,\, q(T) = \alpha\right\}.$$
Find an extremal point of $\psi$. Is this extremal point a candidate to be a maximum or a minimum?

2. Consider the functional $\psi : K \to \mathbb{R}$ defined by
$$\psi(\gamma) := \int_0^1 \sqrt{q^2 + \dot{q}^2} \,\mathrm{d}t,$$
where $K$ is the space of all $C^1$-curves
$$\gamma = \left\{(t, q) \,:\, q = q(t),\, q \in C^1([0,1], \mathbb{R}),\, q(0) = 0,\, q(1) = 1\right\}.$$
Write down the Euler-Lagrange equation.
Prove that $\psi(\gamma) > 1$ for all $\gamma \in K$.

3. Consider the functional $\psi : K \to \mathbb{R}$ defined by
$$\psi(\gamma) := \int_1^2 \frac{1}{t}\sqrt{1 + \dot{q}^2} \,\mathrm{d}t,$$
where $K$ is the space of all $C^1$-curves
$$\gamma = \left\{(t, q) \,:\, q = q(t),\, q \in C^1([1,2], \mathbb{R}),\, q(1) = 0,\, q(2) = 1\right\}.$$
Find an extremal point of $\psi$.

TURN OVER

**Exercise 2** (5 pts)

Consider a function $L \in C^2(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ and a functional $\psi : K \to \mathbb{R}$ defined by

$$\psi(\gamma) := \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, \mathrm{d}t,$$

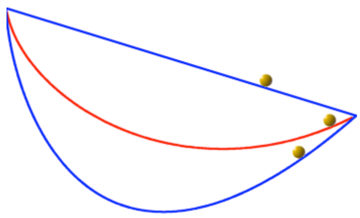
where $K$ is the space of all $C^1$-curves

$$\gamma = \left\{(t, q) \;:\; q = q(t),\; q \in C^1([t_1, t_2], \mathbb{R}),\; q(t_1) = q_1,\; q(t_2) = q_2\right\}.$$

Assume that the corresponding Euler-Lagrange equation is identically satisfied for any $\gamma$. Prove that $\psi$ does not depend on $\gamma$ but only on $(t_1, q_1)$ and $(t_2, q_2)$.

**Exercise 3** (2+3+2+2+1 pts)

We look for the optimal shape of a wire that connects two fixed points $A$ and $B$ on a vertical plane. A bead of unit mass falls along this wire, without friction, under the influence of gravity. The shape of the wire is defined to be optimal if the bead falls from $A$ to $B$ in as short a time as possible.
Let $y = y(x)$ be the function which describes the shape of the wire on the $(x, y)$-plane, connecting $A := (0, 0)$ and $B := (a, b)$ with $a > 0$ and $b \geq 0$. We assume that the positive $y$-axis is pointing downward.



The associated falling time is

$$T(y) = \frac{1}{\sqrt{2\,g}} \int_0^a \sqrt{\frac{1 + (y')^2}{y}}\, \mathrm{d}x, \qquad y' := \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Here $g$ is the constant gravitational acceleration (you can fix $g = 1/2$). To solve the problem one has to minimize the functional $T$ over the set of all functions $y \in C^1([0, a], [0, \infty))$ with $(y(0), y(a)) = (0, b)$.

1. Justify, using elementary mechanics, the formula for $T(y)$.
2. Construct the Euler-Lagrange equation of the problem.
3. Reduce this second-order ODE to a first-order ODE of the form
   $$y\big(1 + (y')^2\big) = c,$$
   where $c \in \mathbb{R}$ is a constant of integration.
4. Introduce the angular variable $\varphi$, which measures the angle that the tangent to the curve makes with the vertical. Find the family of parametric equations for the plane curve $y = y(x)$ which minimizes $T$ ($\varphi$ is the parameter).
5. Sketch a minimizing curve.