Technische Universität Berlin
Institut für Mathematik

Mathematical Physics I, Dr. Matteo Petrera, Mats Vermeeren — WS 17/18

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/MP1/

Exercise Sheet 8 (14.12.17)

Due date: ***11.01.18***

- To get the Übungsschein (necessary condition for the oral exam) you need to obtain an *average of 60% in each half of the semester*, i.e. on the first 7 sheets and on the remaining 6 or 7 sheets. Each exercise sheet has 25 points.
- Please work in fixed groups of *2 students*.
- Justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* You can write your answers in English or in German.
- Please turn in your homework at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (2+2+2 pts)

Consider a point with mass $m = 1$ moving in $\mathbb{R}^3$ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) + \alpha(q_1\dot{q}_2 - q_2\dot{q}_1), \qquad \alpha > 0.$$

1. Write down the Euler-Lagrange equations
2. Show that the system is invariant under rotations about the q_3-axis
3. Use the Noether Theorem to find the integral of motion corresponding to the above symmetry. Verify explicitly that the integral of motion is a conserved quantity.

Exercise 2 (3+2 pts)

Consider a one-dimensional mechanical system describing the motion of a point (mass $m = 1$) under the influence of a potential energy

$$U(q) := \frac{q^4}{\alpha + \beta\, q^2},$$

where α, β are parameters.

1. Determine the values of α and β for which one can decide by using the Dirichlet Theorem that the origin $q = 0$ is a stable fixed point.
2. Linearize the dynamical system around this stable fixed point and determine the frequency $\omega = \frac{2\pi}{T}$ of small oscillations, where T is the period of the oscillation.

TURN OVER

Exercise 3 (1+3+2 pts)

In the Hamiltonian phase space $\mathbb{R}^6$ consider the Hamiltonian

$$\mathcal{H}(q,p) := \sqrt{\alpha^2 + \langle p, p \rangle} + \langle b, q \rangle, \qquad \alpha > 0,$$

where $b \in \mathbb{R}^3$ is a constant vector.

1. Derive the Hamilton equations.
2. Construct the Lagrangian.
3. Derive the Euler-Lagrange equations.

Exercise 4 (2+2+1 pts)

In $\mathbb{R}$ consider the system of N Newton

$$\ddot{q}_k = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}},$$

where $k = 1, \ldots, N$ and $q_{N+k} \equiv q_k \pmod{N}$.

1. Prove that the above equations of motion are Euler-Lagrange equations for the Lagrangian

$$\mathcal{L}(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N) := \sum_{k=1}^{N} \left(\frac{\dot{q}_k^2}{2} - e^{q_{k+1}-q_k} \right).$$

2. Construct the Hamiltonian of the system and write down the Hamilton equations.
3. Prove that the total linear momentum

$$P(p_1, \ldots, p_N) := \sum_{k=1}^{N} p_k$$

is an integral of motion.

Exercise 5 (2+1 pts)

Consider the following IVP in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = \operatorname{grad}_x G(x), \\ x(0) \in \mathbb{R}^2, \end{cases}$$

where $G \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$.

1. Under which conditions is the flow of this IVP a canonical Hamiltonian flow?
2. Under those conditions, determine the Hamiltonian.