1 Consider the IVP $\qquad \dot{x} = x - \frac{4}{x}$

$$x(0) = -1$$

(a) Find the solution

(b) Determine the maximal interval where the solution exists, is unique and is C^1.

(a) $\dot{x} = \frac{x^2-4}{x} \quad \Rightarrow \quad \dot{x}\,dt = \frac{x^2-4}{x}\,dt$

$$\Rightarrow \frac{x\,dx}{x^2-4} = dt$$

$$\Rightarrow \int_{x(0)}^{x} \frac{y}{y^2-4}\,dy = \int_0^t ds$$

$$\Rightarrow \int_{x(0)^2-4}^{x^2-4} \frac{\frac{1}{2}dz}{z} = t \qquad \begin{array}{l} z = y^2-4 \\ dz = 2y\,dy \end{array}$$

$$\Rightarrow \frac{1}{2}\log|z| \Big|_{x(0)^2-4}^{x^2-4} = t$$

$$\Rightarrow \frac{1}{2}\log|x^2-4| - \frac{1}{2}\log 3 = t$$

At $x^2-4=0$ the solution is not defined, so the solution never changes sign. Since $x(0)^2-4=-3$, we will always have $x^2-4<0$

$$\Rightarrow \log(4-x^2) - \log 3 = 2t$$

$$\Rightarrow 4 - x^2 = 3e^{2t}$$

$$\Rightarrow x = \pm\sqrt{4-3e^{2t}}.$$

Since $x(0)<0$, we need the $-$ sign:

$$x = -\sqrt{4-3e^{2t}}$$

(b) Away from $x=0$ the rhs $\frac{x^2-4}{x}$ is C^1, so the solution ~~should be~~ is unique where it exists.

For the formula to make sense we need

$$4-3e^{2t} \geq 0 \iff -e^{2t} \geq -\frac{4}{3}$$
$$\iff e^{2t} \leq \frac{4}{3}$$
$$\iff 2t \leq \log \frac{4}{3}$$
$$\iff t \leq \frac{1}{2} \log \frac{4}{3}$$

So the solution exists on $(-\infty, \frac{1}{2}\log\frac{4}{3}]$.

What happens at the endpoint?

The rhs $\frac{x^2-4}{x}$ is not defined

$\rightarrow$ no way to extend solution by different formula

$\Rightarrow (-\infty, \frac{1}{2}\log\frac{4}{3}]$ is the maximal interval of existence.

2 Solve the IVP $\quad \dot{x} = \frac{1}{2}(x^2 - 1)$

$$x(0) = x_0$$

and find its maximal interval of existence. What happens at the boundaries of this interval?

If $x_0 = 1$ or $x_0 = -1$ we find a constant solution, $x = x_0$.

Assume $x \neq \pm 1$.

$$\frac{dx}{x^2 - 1} = \frac{1}{2}\,dt \implies \int_{x_0}^{x} \frac{dy}{y^2 - 1} = \frac{1}{2} t$$

$$\implies \int_{x_0}^{x} \left(\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1} \right) dy = \frac{1}{2} t$$

$$\implies \ln|x-1| - \ln|x+1| - \ln|x_0 - 1| + \ln|x_0 + 1| = t$$

$$\implies \ln\left|\frac{x-1}{x+1}\right| = t + \ln\left|\frac{x_0 - 1}{x_0 + 1}\right|$$

$$\implies \frac{x-1}{x+1} = \pm e^t \, \frac{x_0 - 1}{x_0 + 1}$$

We take the + sign, since at $t=0$ we must have $\frac{x-1}{x+1} = \frac{x_0 - 1}{x_0 + 1}$.

We find

$$x - 1 = \left(e^t \frac{x_0 - 1}{x_0 + 1} \right)(x + 1)$$

$$\implies x = \frac{1 + e^t \frac{x_0 - 1}{x_0 + 1}}{1 - e^t \frac{x_0 - 1}{x_0 + 1}}$$

For which t is this defined?

We have 3 cases:

(I) $x_0 \in (-1, 1)$

Then $x_0 - 1 < 0$ and $x_0 + 1 > 0$, so $\frac{x_0-1}{x_0+1} < 0$.

~~In particular~~ It follows that $1 - e^t \frac{x_0-1}{x_0+1} \neq 0$ for all $t \in \mathbb{R}$

The solution is defined everywhere.

Let $a := -\frac{x_0-1}{x_0+1} > 0$. Then

$$x(t) = \frac{1-ae^t}{1+ae^t} \leq 1$$

and

$$\lim_{t\to\infty} x(t) = \frac{-a}{a} = -1$$

$$\lim_{t\to-\infty} x(t) = \frac{1}{1} = 1$$

(II) $x_0 > 1$

Then $\frac{x_0-1}{x_0+1} > 0$, so the denominator reaches 0 when

$$e^t \frac{x_0-1}{x_0+1} = 1 \iff t = \log \frac{x_0+1}{x_0-1} > 0$$

The solution is defined on $\left(-\infty, \log \frac{x_0+1}{x_0-1}\right)$.

The limits are

$$\lim_{t\to-\infty} x = 1 \quad \text{(as before)}$$

$$\lim_{t \nearrow \log \frac{x_0+1}{x_0-1}} x = +\infty \quad \text{(since both numerator and denominator } > 0\text{)}$$

(III) $x_0 < -1$ Now even > 1

Again $\frac{x_0-1}{x_0+1} > 0$. The denominator is zero when

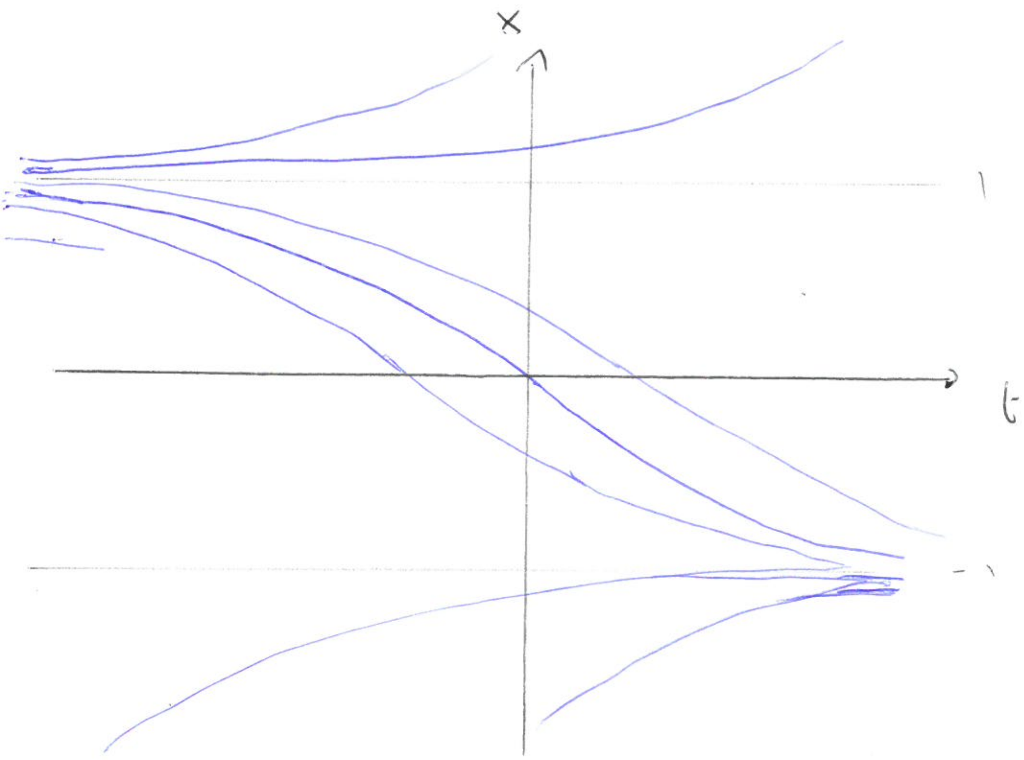
$$t = \log \frac{x_0+1}{x_0-1}$$

as before, but now $\log \frac{x_0+1}{x_0-1} < 0$, so the interval of existence is $\left(\log \frac{x_0+1}{x_0-1}, +\infty\right)$.

The limits are

$\lim_{t \to +\infty} x = -1$ as in case (I)

$\lim_{t \to \log \frac{x_0+1}{x_0-1}} x = -\infty$ (numerator > 0, denominator < 0)



3. Show that all solutions of the ODE

$$\dot{x} = 1 + x^{14}$$

diverge to infinity in finite time. (Do not solve exactly).

$$\frac{dx}{1+x^{14}} = dt \quad \Rightarrow \quad \int_{x_0}^{x} \frac{dy}{1+y^{14}} = t$$

$$\Rightarrow t \leqslant \int_{-\infty}^{+\infty} \frac{dy}{1+y^{14}} \qquad \text{(integrand } >0\text{)}$$

$$\cancel{\leqslant} \; 2 \cdot \int_0^1 \frac{dy}{1+y^{14}} + 2\int_1^{\infty} \frac{dy}{1+y^{14}} \qquad \text{(integrand even fcn)}$$

$$\leqslant 2 + 2\int_1^{\infty} \frac{dy}{1+y^2} \qquad (1+y^2 \leq 1+y^{14} \text{ for } y \geq 1)$$

$$= 2 + 2\left(\arctan y \Big|_1^{\infty}\right)$$

$$= 2 + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 2 + \frac{\pi}{2}$$

$\rightsquigarrow$ whatever the value of x, ~~theorem~~ it is reached by the solution for some $t \leqslant 2 + \frac{\pi}{2}$

That means that the solution diverges to ∞ before that.

What is special about the number 14?

If $\dot{x} = f(x)$ for some polynomial f of degree >1, solutions will blow up in finite time.