26.10.17

Which of the following are a flow of a dynamical system? (Form a 1-parameter Lie group of diffeomorphisms?)

1. $\phi_t(x) = x+t$
2. $\phi_t(x) = (1+t)x$
3. $\phi_t(x) = e^t x$
4. $\phi_t(x) = e^{t^2} x$
5. $\phi_t(x) = e^{tx}$
6. $\phi_t(x) = \sin(x+t)$
7. $\phi_t(x) = x \cos t$

Check $\phi_{t+s}(x) = (\phi_t \circ \phi_s)(x)$ $\quad s,t \in \mathbb{R}$

1. $\phi_{t+s}(x) = x+t+s = (x+s)+t$ ✓
2. $\phi_{t+s}(x) = (1+t+s)x \neq (1+t)((1+s)x)$ ✗
3. $e^{t+s}x = e^t e^s x$ ✓
4. $e^{(t+s)^2}x \neq e^{t^2} e^{s^2} x$ ✗
5. $e^{(t+s)x} \neq e^{te^{sx}}$ ✗ (also $\phi_0(x) = 1 \neq x$)
6. $\sin(x+t+s) \neq \sin(\sin(x+s)+t)$ ✗ (also $\phi_0(x) \neq x$)
7. $x\cos(t+s) \neq x\cos(s)\cos(t)$ ✗

Invertibility follows from setting $s=-t$ and $\phi_0(x) = x$.

The fact that $\phi_0(x) = x$ should be checked!

Prove that

$$\Phi_t : (x_1, x_2) \mapsto \left(x_1 + t, \frac{x_1 x_2}{x_1 + t}\right)$$

defines a 1-parameter Lie group of diffeomorphisms

Find the infinitesimal generator.

Write down the IVP.

- $$\Phi_{t+s}(x_1, x_2) = \left(x_1 + (t+s), \frac{x_1 x_2}{x_1 + (t+s)}\right)$$

$$(\Phi_t \circ \Phi_s)(x_1, x_2) = \Phi_t\left(x_1 + s, \frac{x_1 x_2}{x_1 + s}\right)$$

$$= \left((x_1+s)+t, \frac{(x_1+s)\frac{x_1 x_2}{x_1+s}}{(x_1+s)+t}\right)$$

$$= \left(x_1 + s + t, \frac{x_1 x_2}{x_1 + s + t}\right) = \Phi_{t+s}(x_1, x_2) \qquad \checkmark$$

- $$\Phi_0(x_1,x_2) = \left(x_1, \frac{x_1 x_2}{x_1}\right) = (x_1, x_2) \qquad \checkmark$$

- $$(\Phi_t \circ \Phi_{-t})(x_1, x_2) = \Phi_t\left(x_1 - t, \frac{x_1 x_2}{x_1 - t}\right)$$

$$= \left(x_1, \frac{(x_1-t)\frac{x_1 x_2}{x_1 - t}}{(x_1 - t)+t}\right) = (x_1, x_2) \qquad \checkmark$$

$$\left.\frac{d\Phi_t}{dt}\right|_{t=0} = \left(1, \frac{-x_1 x_2}{(x_1)^2}\right),$$ so the infinitesimal generator

is

$$\frac{\partial}{\partial x_1} - \frac{x_2}{x_1}\frac{\partial}{\partial x_2}$$

and the corresponding IVP is

$$\begin{cases} \dot{y}_1 = 1 & y_1(0) = x_1 \\ \dot{y}_2 = -\frac{y_2}{y_1} & y_2(0) = x_2 \end{cases}$$

Discrete dynamical system defined by iteration of

$\phi : \mathbb{C} \to \mathbb{C} : \quad z \mapsto e^{2\pi i \alpha} z, \qquad \alpha \in \mathbb{R}.$

Are there periodic orbits?

If $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$, then

$$\phi^q(z) = e^{q \cdot 2\pi i \cdot \frac{p}{q}} z = e^{2p\pi i} z = z,$$

so every orbit is periodic.

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there are no periodic orbits.

Proof by contradiction:

Assume $\phi^n(z) = z$

$\Rightarrow e^{2n\pi i \alpha} = 1$

$\Rightarrow n\alpha \in \mathbb{Z}$

$\Rightarrow \alpha = \frac{m}{n}$ for some $m \in \mathbb{Z}$ $\lightning$.

Orbit of z: $\mathcal{O}(z) = \{ \phi^n(z) \mid n \in \mathbb{Z} \}$

$= \{ e^{2n\pi i \alpha} z \mid n \in \mathbb{Z} \}$

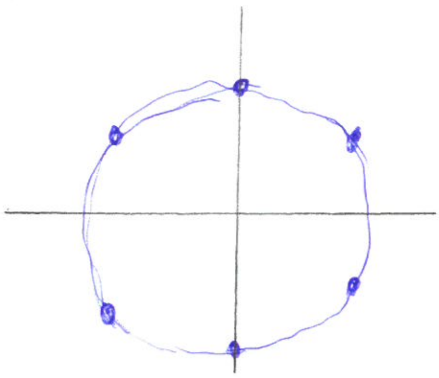
Are there invariant sets / invariant functions ?

The function $f: \mathbb{C} \to \mathbb{R}^+ : f(z) = |z|$ is invariant,

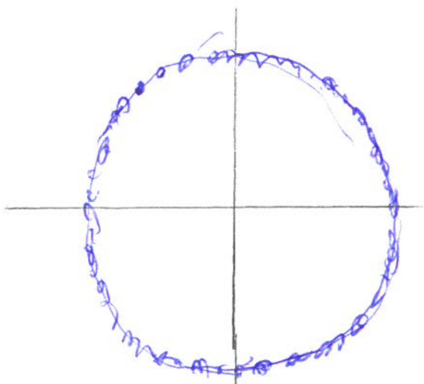
so all circles centered at the origin

$$\{ z \in \mathbb{C} \mid |z| = R \} \qquad R \in \mathbb{R}^+$$

are invariant sets.



$\alpha \in \mathbb{Q}$: periodic



$\alpha \in \mathbb{R} \setminus \mathbb{Q}$: non-periodic, dense in invariant set.

Fixed points ? $z = 0$

(All $z \in \mathbb{C}$ if $\alpha \in \mathbb{Z}$)

Does the IVP $\begin{cases} \dot{x} = -\operatorname{sgn} x \\ x(0) \in \mathbb{R} \end{cases}$ define a dynamical system?

Yes. The flow is

$$\phi_t(x_0) = \begin{cases} x_0 - t & \text{if } x_0 > t \geq 0 \\ 0 & \text{if } t > |x_0| \\ x_0 + t & \text{if } -x_0 > t \geq 0 \end{cases} \qquad \text{for } t \geq 0$$

$$(\phi_t \circ \phi_s)(x_0) = \begin{cases} \phi_t(x_0 - s) & x_0 > s > 0 \\ \phi_t(0) & s > |x_0| \\ \phi_t(x_0 + s) & -x_0 > s > 0 \end{cases}$$

$$= \begin{cases} (x_0 - s - t) & x_0 - s > t > 0 \\ 0 & s + t > |x_0| \\ x_0 + s + t & \text{if } -x_0 - s > t > 0 \end{cases}$$

$$= \phi_{t+s}(x_0)$$

The flow is <u>not</u> invertible: at time T

$$\phi_T(x_0) = 0$$

for all initial conditions with $|x_0| \leq T$.

Does iteration of the map

$$\phi : \mathbb{N} \to \mathbb{N} \; ; n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

define a discrete dynamical system?

Yes, but it is not invertible, e.g.

$1 \mapsto 4$

but also

$8 \mapsto 4$

Collatz conjecture: for all initial values, the iteration will end up in the cycle $1 \mapsto 4 \mapsto 2 \mapsto 1$

Consider the dynamical system on $\mathbb{R}$ described by

$$\dot{x} = -x$$

Find (i) its fixed points

(ii) its invariant sets

(iii) its forward invariant sets

(i) $\dot{x} = 0 \iff x = 0$

All solutions converge to 0, so it is an asymptotically stable fixed point.

(ii) $\{0\}$, $[0, \infty)$, $(0, \infty)$, $(-\infty, 0)$, $(-\infty, 0]$, $\mathbb{R}$

(iii) all intervals containing 0, or with 0 as a boundary.

Are they stable, attracting, asymptotically stable?

yes | iff boundary is not open at 0