

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 1

(Manifolds)

due 30.10.2018

Exercise 1

3 points

Let X be a topological space, $x \in X$ and $n \geq 0$. Show that the following statements are equivalent:

- i) There is a neighborhood of x which is homeomorphic to \mathbb{R}^n .
- ii) There is a neighborhood of x which is homeomorphic to an open subset of \mathbb{R}^n .

Exercise 2

6 points

Let $n \in \mathbb{N}$ and $X := \mathbb{R}^{n+1} \setminus \{0\}$. The quotient space $\mathbb{R}P^n = X/\sim$ with equivalence relation given by

$$x \sim y \iff x = \lambda y, \quad \lambda \in \mathbb{R}$$

is called the n -dimensional *real projective space*. Let $\pi: X \rightarrow \mathbb{R}P^n$ denote the *canonical projection* $x \mapsto [x]$. For $i = 0, \dots, n$, we define $U_i := \pi(\{x \in X \mid x_i \neq 0\})$ and $\varphi_i: U_i \rightarrow \mathbb{R}^n$ by

$$[x_0, \dots, x_n] \mapsto (x_0/x_i, \dots, \widehat{x_i}, \dots, x_n/x_i).$$

Show that

- a) π is an *open map*, i.e. maps open sets in X to open sets in $\mathbb{R}P^n$,
- b) the maps φ_i are well-defined and $\{(U_i, \varphi_i)\}_{i \in I}$ is a smooth atlas of $\mathbb{R}P^n$,
- c) $\mathbb{R}P^n$ is compact. **Hint:** Note that the restriction of π to S^n is surjective.

Exercise 3

6 points

Let M and N be topological manifolds of dimension m and n , respectively. Show that their Cartesian product $M \times N$ is a topological manifold of dimension $m + n$. Show further that, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is a smooth atlas of M and $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ is a smooth atlas of N , then $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{(\alpha, \beta) \in A \times B}$ is a smooth atlas of $M \times N$. Here $\varphi_\alpha \times \psi_\beta: U_\alpha \times V_\beta \rightarrow \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta)$ is defined by $\varphi_\alpha \times \psi_\beta(p, q) := (\varphi_\alpha(p), \psi_\beta(q))$.