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http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/DGII/ WS 18

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 5

(Lie bracket, vector bundles, connections)

due 27.11.2018

Exercise 1 **5 points**

Let $f\colon M \to N$ be a diffeomorphism. For a vector field $X \in \Gamma(TM)$ the push forward $f_*X \in \Gamma(TN)$ of X is defined by $f_*X := df \circ X \circ f^{-1}$. Show that, for $X, Y \in \Gamma(TM)$,

$$f_*[X, Y] = [f_*X, f_*Y]\,.$$

Exercise 2 **5 points**

Show that the tangent bundle $T\mathbb{S}^3$ of the round sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ is trivial.

Hint: Show that the vector fields $\varphi_1(x_1, x_2, x_3, x_4) = (-x_2, x_1, x_4, -x_3)$, $\varphi_2(x_1, x_2, x_3, x_4) = (x_3, x_4, -x_1, -x_2)$ and $\varphi_3(x_1, x_2, x_3, x_4) = (-x_4, x_3, -x_2, x_1)$ form a frame of $T\mathbb{S}^3$.

Exercise 3 **5 points**

Let ∇ be a connection on a direct sum $E = E_1 \oplus E_2$ of two vector bundles over M. Show that

$$\nabla = \begin{pmatrix} \nabla^1 & A \\ \tilde{A} & \nabla^2 \end{pmatrix},$$

where $\tilde{A} \in \Omega^1(M, \mathrm{Hom}(E_1, E_2))$, $A \in \Omega^1(M, \mathrm{Hom}(E_2, E_1))$ and ∇^i are connections on the bundles E_i.