TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik
Pinkall / Knöppel
http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/DGII/ WS 18

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 6

(Connections and differential forms)

due 4.12.2017

Exercise 1 — 5 points

Let $M \subset \mathbb{R}^2$ be open. On $E = M \times \mathbb{R}^2$ we define two connections ∇ and $\tilde{\nabla}$ as follows:

$$\nabla = d + \begin{pmatrix} 0 & -x\,dy \\ x\,dy & 0 \end{pmatrix}, \quad \tilde{\nabla} = d + \begin{pmatrix} 0 & -x\,dx \\ x\,dx & 0 \end{pmatrix}.$$

Show that (E, ∇) is not trivial. Further construct an explicit isomorphism between $(E, \tilde{\nabla})$ and the trivial bundle (E, d).

Exercise 2 — 5 points

Let $M = \mathbb{R}^3$. Determine which of the following forms are closed ($d\omega = 0$) and which are exact ($\omega = d\theta$ for some θ):

a) $\omega = yz\,dx + xz\,dy + xy\,dz$,
b) $\omega = x\,dx + x^2y^2\,dy + yz\,dz$,
c) $\omega = 2xy^2\,dx \wedge dy + z\,dy \wedge dz$.

If ω is exact, please write down the potential form θ explicitly.

Exercise 3 — 5 points

Let $M = \mathbb{R}^n$. For $\xi \in \Gamma(\mathrm{T}M)$, we define $\xi^\flat \in \Omega^1 M$ and $\iota_\xi \det \in \Omega^{n-1} M$ as follows:

$$\xi^\flat(X_1) := \langle \xi, X_1 \rangle, \ \iota_\xi \det(X_2, \ldots, X_n) := \det(\xi, X_2, \ldots, X_n), \ X_1, \ldots, X_n \in \Gamma(\mathrm{T}M).$$

Show the following identities:

$$df = (\operatorname{grad} f)^\flat, \quad d(\iota_\xi \det) = \operatorname{div}(\xi) \det,$$

and for $n = 3$,

$$d(\xi^\flat) = \iota_{\operatorname{curl} \xi} \det.$$