Technische Universität Berlin
Institut für Mathematik
Pinkall / Knöppel
http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/DGII/ WS 18

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 7

(Orientation, Stokes theorem)

due 11.12.2017

Exercise 1 **5 points**

For which $n \in \mathbb{N}$ is $\mathbb{R}\mathrm{P}^n$ orientable, and for which not? Prove it.

Exercise 2 **5 points**

Prove that an n-dimensional manifold M for which there exists an immersion $f\colon M \to \mathbb{R}^{n+1}$ is orientable if and only if there is a smooth nowhere-vanishing normal vector field along f, i.e. a map $N\colon M \to \mathbb{S}^n$ such that $N(p) \perp d_pf\,(T_pM)$ for all $p \in M$.

Exercise 3 **5 points**

Consider $\mathbb{R}^n$ with standard metric $\langle .,. \rangle$. Let $M \subset \mathbb{R}^n$ be an n-dimensional submanifold with boundary with outward pointing normal vector field $N\colon \partial M \to \mathrm{S}^{n-1} \subset \mathbb{R}^n$. Then the volume form of M is given by $\omega_M = \det$ and the induced volume form on ∂M is given by $\omega_{\partial M} = \iota_N \omega_M$. Let $X \in \Gamma(T\mathbb{R}^n)$. Show that

$$\int_M \operatorname{div}(X)\, \omega_M = \int_{\partial M} \langle X, N \rangle\, \omega_{\partial M}\,.$$

Conclude that, if A denotes the area of an n-dimensional sphere of radius r and V the volume enclosed by it, then $(n+1)V = rA$.