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http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/DGII/ WS 18

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 10

(Geodesics, parallel transport)

due 15.1.2019

Exercise 1 **5 points**

a) Is there a Riemannian manifold (M, g) which has finite diameter (i.e. there is an m such that all points $p, q \in M$ have distance $d(p, q) < m$) and there is a geodesic of infinite length without self-intersections?

b) Find an example for a Riemannian manifold diffeomorphic to $\mathbb{R}^n$ but which has no geodesic of infinite length.

Exercise 2 **5 points**

Let $\gamma\colon I \to M$ be an arc-length parameterized curve. Show that γ is a geodesic if and only if γ is a locally shortest, i.e. for all $t \in I$ exists an $\epsilon > 0$ such that for all $t_1, t_2 \in [t - \epsilon, t + \epsilon]$ the restriction $\gamma|_{[t-\epsilon,t+\epsilon]}$ is the shortest path between $\gamma(t_1)$ and $\gamma(t_2)$. **Hint**: Gauß' lemma.

Exercise 3 **5 points**

Let γ be a curve in a smooth manifold M and $\gamma(t_0)$ a point on it. The mapping $P_{\gamma(t),\gamma(t_0)}\colon T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ defined by $P_{\gamma(t),\gamma(t_0)}X_{\gamma(t_0)} = X_{\gamma(t)}$, where $X_{\gamma(t)}$ is the unique extension of $X_{\gamma(t_0)}$ to a parallel vector field along γ, is called the parallel transport from $X_{\gamma(t_0)}$ to $X_{\gamma(t)}$. Show:

a) The parallel transport is a linear isomorphism, and if $X(t) = X_{\gamma(t)}$ is a vector field along γ, then

$$(\gamma^*\nabla)_{\frac{\partial}{\partial t}} X \Big|_{t=t_0} = \lim_{t\to t_0} \tfrac{1}{t-t_0}\big(P_{\gamma(t_0),\gamma(t)}X(t) - X(t_0)\big).$$

b) If M is an oriented Riemannian manifold with Riemannian connection ∇, then $P_{\gamma(t),\gamma(t_0)}$ is an orientation-preserving isometry.