

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 12

(Conjugate Points, Synge's lemma, Hadamard-Cartan theorem)

due 05.02.2018

Definition

Let (M, g) be a Riemannian manifold and $\gamma: [a, b] \rightarrow M$ be a geodesic with end points p and q . The points p and q are said to be *conjugate along* γ if there exists a non-trivial Jacobi field Y along γ such that $Y(a) = 0 = Y(b)$.

Exercise 1

5 points

Let $\gamma: [a, b] \rightarrow M$ be a geodesic in a Riemannian manifold (M, g) . If there exists $s_0 \in (a, b)$ such that $\gamma(a)$ and $\gamma(s_0)$ are conjugate along γ , then there is a variation γ_t of γ with fixed end points such that for t small enough

$$L(\gamma_t) < L(\gamma) \text{ and } E(\gamma_t) < E(\gamma).$$

Hint: Construct a (piecewise smooth) vector field along γ such that the second variation is negative.

Exercise 2

5 points

Let M be an even-dimensional orientable Riemannian manifold of strictly positive sectional curvature and let γ be a closed geodesic, i.e. $\gamma: \mathbb{S}^1 \rightarrow M$ such that $\gamma'' = 0$, of length $L(\gamma) > 0$. Show that there is a length-decreasing variation of γ .

Hint: Show that there exists a non-trivial parallel vector field along γ .

Exercise 3

5 points

Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature. Show that for every $m \in M$ there is a metric \tilde{g} on $T_m M$ such that \exp_m is a local isometry and that $(T_m M, \tilde{g})$ is complete.