Exercise Sheet 3

Exercise 1: Sphere intersections. (4 pts)
A non-empty intersection of two planes in $\mathbb{R}^3$ is a line. Prove that a non-empty intersection of two spheres in $\mathbb{R}^3$ is a circle (considering a point as a circle of radius 0).

Exercise 2: Stereographic projection in $\mathbb{R}^3$. (5 pts)
Give a brief description and provide a sketch for each of the following questions. Feel free to use colors!

(a) What are the images of the lines of latitude (the circles centered at $\pm e_3$) and of the lines of longitude (the circles through $\pm e_3$) under stereographic projection?

(b) Now rotate the pattern on the sphere to consider the circles centered at $\pm e_1$ along with the great circles through $\pm e_1$. What are their images under stereographic projection?

(c) Consider the horizontal lines in the plane (lines of the form $y = k$). What are their preimages under stereographic projection?

Exercise 3: Stereographic projection in $\mathbb{R}^2$. (4 pts)
If $C \subset \mathbb{R}^2$ is the circle centered at $p$ with radius $r$, then inversion in $C$ is the map $\tau$ from $\mathbb{R}^2 \setminus \{p\}$ to itself sending any point $x$ to the point $\tau(x)$ along the same ray from $p$ such that $\|\tau(x) - p\| = r^2/\|x - p\|$. Now consider the stereographic projection

$\sigma : S^1 \setminus \{(0, 1)\} \to \mathbb{R}$, \hspace{1cm} $(x_1, x_2) \mapsto \frac{x_1}{1 - x_2}$.

Find a circle $C \subset \mathbb{R}^2$ such that $\sigma$ is the restriction of the inversion in $C$.

Exercise 4: Diagonalizing quadratic forms. (3 pts)
Consider the quadratic form $Q(x)$ on $\mathbb{R}^3$ defined by $Q(x) = x_1x_2 + x_2x_3 + x_3x_1$. Find a linear change of coordinates $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that with respect to the coordinates $y = T(x)$, the quadratic form is diagonal: $Q(x) = \sum_{i=1}^3 y_i^2$. What is its signature?

Due: Before lecture on Wednesday, 14.11.2018.