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Geometry I - WS 18

http://www.math.tu-berlin.de/geometrie/Lehre/WS18/GeometryI/

Exercise Sheet 4

Exercise 1. (6 pts)

Consider the orthogonal group $O(1,1)$ acting on the Lorentz space $\mathbb{R}^{1,1}$ with its scalar product $\langle x, y\rangle = x_1y_1 - x_2y_2$.

(a) Show that any v with $\langle v, v\rangle = 1$ can be written as $v = (\pm\cosh(t), \sinh(t))$ for some $t \in \mathbb{R}$, while any v with $\langle v, v\rangle = -1$ can be written as $v = (\sinh(t), \pm\cosh(t))$.

(b) Consider the family of matrices

$$R_t := \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \in O(1,1).$$

Show that for any $s, t \in \mathbb{R}$ we have $R_sR_t = R_{s+t}$.

(c) Show that every matrix in $O(1,1)$ can be written as DR_t where D is one of the four possible diagonal matrices with diagonal entries ±1 and R_t is as above for some $t \in \mathbb{R}$.

Exercise 2. (4 pts)

Let V be the four-dimensional vector space of real 2×2 matrices.The determinant defines a quadratic form $Q(M) := \det M$ on V. Find the associated symmetric bilinear form $B(M,N)$ on V (such that $Q(M) = B(M,M)$). Find a basis for V so that the matrix of B is in *standard form*: it is diagonal and the diagonal entries are elements of the set $\{1, -1, 0\}$, in that order. What is the signature of B?

Exercise 3. (4 pts)

Given a subset $S \subseteq \mathbb{R}^{n,1}$ in Lorentz space, let $S^\perp$ denote its orthogonal complement

$$S^\perp := \{x \in \mathbb{R}^{n,1} | \langle x, y\rangle = 0 \ \forall y \in S\}.$$

(a) Show that $S^\perp$ is a vector subspace.
(b) Let $\mathbb{L} \subseteq \mathbb{R}^{n,1}$ denote the light cone. What is $\mathbb{L}^\perp$?
(c) Prove that, for any $S \subseteq \mathbb{R}^{n,1}$, $S \subseteq (S^\perp)^\perp$.
(d) Does equality hold in the above?

Exercise 4. (2 pts)

Prove that the columns of an orthogonal matrix in $O(p,q)$ form an orthonormal basis with respect to the scalar product on $\mathbb{R}^{p,q}$.

Due: Before lecture on Wednesday, 21.11.2018.