TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I - WS 18

http://www.math.tu-berlin.de/geometrie/Lehre/WS18/GeometryI/

Exercise Sheet 5

Let $\langle\ ,\ \rangle$ denote the Lorentz inner product on $\mathbb{R}^{2,1}$.

Exercise 1: Hyperbolic circles. (4 pts)

Define the circle of radius $r > 0$ around point $c \in \mathbb{H}^2$ as

$$C_r(c) := \{x \in \mathbb{H}^2 \mid d(c,x) = r\},$$

where $d(\cdot,\cdot)$ denotes the hyperbolic distance. Find the length of $C_r(c)$.

Hint: It is enough to consider circles with center $(0,0,1)$. Why?

Exercise 2: Orthogonal lines. (4 pts)

Let l_1 and l_2 be hyperbolic lines with normals n_1 and n_2. Show that there exists a unique hyperbolic line l_3 such that $l_1 \perp l_3$ and $l_2 \perp l_3$ if and only if $|\langle n_1, n_2\rangle| > 1$.

Exercise 3: Angle of parallelism. (4 pts)

Consider hyperbolic right triangles ABC with a right angle at C. Denote the side lengths by a, b, c and the interior angles by $\alpha, \beta, \pi/2$ as usual.

(a) Show that $\tanh(a) = \sinh(b)\tan(\alpha)$.
(b) Now suppose we fix b and consider a family of triangles with $a \to \infty$. Show that for these triangles $\alpha \to 2\tan^{-1}(e^{-b})$.

Exercise 4: Lines in the light cone. (4 pts)

Let $n \in \mathbb{R}^{2,1}$ be a unit space-like vector (that is, $\langle n, n\rangle = 1$) and let $U = \{v \in \mathbb{R}^{2,1} \mid \langle v, n\rangle = 0\}$ be the plane Lorentz-orthogonal to n. Denote by $\mathcal{L} \subseteq \mathbb{R}^{2,1}$ the light cone.

(a) Show that $\mathcal{L} \cap U$ consists of two lines l_1 and l_2.
(b) Show that the span of n and l_1 (resp. n and l_2) is a plane in $\mathbb{R}^{2,1}$ tangent to $\mathcal{L}$.

Due: Before lecture on Wednesday, 28.11.2018.