TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. Boris Springborn
Hana Kouřimská

Geometry I - WS 18

http://www.math.tu-berlin.de/geometrie/Lehre/WS18/GeometryI/

Exercise Sheet 6

Exercise 1: Hyperbolic distance. (4 pts)

Let $O = (0,0)$ and $P = (x,0)$, for some $0 < x < 1$ be points in the unit disc D. If we interpret D as the Beltrami-Klein model of $\mathbb{H}^2$, show that the hyperbolic distance $d = d(O,P)$ satisfies $x = \tanh(d)$. Interpreting D instead as the Poincaré disc model, show that $x = \tanh(\frac{d}{2})$.

Exercise 2: Hyperbolic circles. (3 pts)

Show that there are circles in the hyperbolic plane which cannot be inscribed in a triangle. More precisely, show that circles with diameter greater than $\ln(3)$ cannot be inscribed in a triangle.

Exercise 3: Regular polygons. (4 pts)

A polygon is *regular* if all its sidelengths and internal angles are equal. For which $n \in \mathbb{N}$ do there exist right-angled regular n-gons in the hyperbolic plane? What is their sidelength (as a function of n)?
Hint: You may assume that for each n all regular polygons are congruent.

Exercise 4: Lorentz cross product. (5 pts)

For $x, y \in \mathbb{R}^{2,1}$ the Lorentz cross product $x \times y \in \mathbb{R}^{2,1}$ is defined such that

$$\det[x,y,z] = \langle x \times y, z\rangle \quad \text{for all } \mathbb{R}^{2,1},$$

where $\langle .\, , . \rangle$ is the Lorentz scalar product of $\mathbb{R}^{2,1}$.

(a) Show that $\langle x \times y, z\rangle = \langle y \times z, x\rangle$.
(b) Find an expression for $x \times y$ in coordinates.
(c) Show that the Lorentz cross product satisfies the equation

$$a \times (b \times c) = c\langle a, b\rangle - b\langle a, c\rangle.$$

(d) Use (a) and (c) to show that

$$\langle x \times y, u \times v\rangle = -\det\begin{pmatrix} \langle x,u\rangle & \langle x,v\rangle \\ \langle y,u\rangle & \langle y,v\rangle \end{pmatrix}.$$

(e) Use (d) to show that for space-like linearly independent $x, y \in \mathbb{R}^{2,1}$, $n \in \text{span}\{x,y\}^{\perp}$,

$$\left| \frac{\langle x,y\rangle}{\sqrt{\langle x,x\rangle\langle y,y\rangle}} \right| \leq 1 \quad \text{if and only if} \quad \langle n,n\rangle \leq 0.$$

Due: Before lecture on Wednesday, 5.12.2018.