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Geometry I - WS 18

http://www.math.tu-berlin.de/geometrie/Lehre/WS18/GeometryI/

Exercise Sheet 7

Exercise 1: Inversion in a circle. (4 pts)

An *isometry* is a map that preserves the length of curves. In the Poincaré half-plane model, consider the inversion in the unit circle given by

$$f : p \mapsto \frac{1}{p_1^2 + p_2^2} p.$$

Show that this is an isometry with respect to the Riemannian metric $g_p(v, w) = \frac{1}{p_2^2} \langle v, w \rangle_{\mathbb{R}^2}$.

Hint: Show that for any curve γ in the half-plane model, $\text{length}(\gamma) = \text{length}(f \circ \gamma)$.

Exercise 2: Area of n-gons. (3 pts)

(a) Find the area of a polygon with interieor angles $\alpha_1, \ldots, \alpha_n$.
(b) What is the maximum area a hyperbolic n-gon can attain?
(c) Which n-gons attain this area?

Exercise 3: Equidistant points (4 pts)

Let ℓ be a hyperbolic line, $\delta > 0$ and

$$E = \{p \in \mathbb{H}^2 \mid d(p, l) = \delta\}.$$

(a) Show that in the hyperboloid model, E is obtained as an intersection of $\mathbb{H}^2$ with two affine planes. Find explicit formulas for the two affine planes in form of

$$U = \{x \in \mathbb{R}^{2,1} \mid \langle x, v \rangle = r\} \quad \text{for some } v \in \mathbb{R}^{2,1} \backslash \{0\}, r > 0.$$

(b) Draw a sketch of ℓ and E in the Poincaré disc model and prove that ℓ and E meet at the boundary of the Poincaré disc.

Exercise 4: Isometries and the Klein model (5 pts)

Let $M := \begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}$.

(a) Show that $M \in O(2, 1)$.
(b) Let $\ell = \mathbb{H}^2 \cap (1, 1, 1)^\perp$. Show that the points in ℓ are fixed under the action of M.
(c) Describe the action of M on the Klein model of hyperbolic geometry. Make a sketch.

Due: Before lecture on Wednesday, 12.12.2018.