Exercise Sheet 9

Exercise 1: Projective transformations in $\mathbb{RP}^1$. (6 pts)

Let $Q := \{[0,1], [1,0], [1,2], [2,1]\} \subset \mathbb{RP}^1$. Consider the projective transformations $\tau : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that $\tau(Q) \subseteq Q$.

(a) How many such transformations $\tau$ exist?
(b) Find explicit formulas for these transformations $\tau$.
(c) Can a projective transformation $\mathbb{RP}^1 \to \mathbb{RP}^1$ which is not the identity have three fixed points?
(d) How many fixed points can a projective transformation $\mathbb{RP}^1 \to \mathbb{RP}^1$ have? Give an example for each possible case.

Exercise 4: Skew lines in $\mathbb{RP}^3$. (3 pts)

(a) Prove that, in general, two lines in $\mathbb{RP}^3$ do not intersect. Such lines are called skew lines.
(b) Given three lines which are pairwise skew, prove that there are infinitely many lines which intersect all three lines.

Exercise 3: Projective transformations in $\mathbb{RP}^2$. (3 pts)

Given the four points $A_1 = (1,0,1), A_2 = (0,1,1), A_3 = (0,0,1), A_4 = (1,1,1) \in \mathbb{R}^3$ and four more points $B_1 = (1,0,0), B_2 = (0,1,0), B_3 = A_3, B_4 = A_4 \in \mathbb{R}^3$. Let $\tilde{f}$ be the projective transformation such that $\tilde{f}([A_i]) = [B_i], i \in \{1,2,3,4\}$. Find the element of $PGL(3, \mathbb{R})$ corresponding to $\tilde{f}$.

Exercise 4: Desargues’ theorem in 3D. (4 pts)

Let $P_1, P_2, P_3, P_4$ and $Q_1, Q_2, Q_3, Q_4$ be distinct points in $\mathbb{RP}^3$ so that neither $P_1, P_2, P_3, P_4$ nor $Q_1, Q_2, Q_3, Q_4$ are contained in a plane. Suppose that the four lines $P_iQ_i$ intersect in a common point. Given a pair of corresponding faces, the planes they lie in intersect in a line. Show that all four such lines lie in some common plane.

Additional Christmas Exercise! (1 extra pt)

Simplify the equation such that the text makes sense.

\[ y = \frac{\ln \left( \frac{x}{2} - sa \right)}{r^2} \quad \text{AND A HAPPY NEW YEAR!} \]

Due: Before lecture on Wednesday, 9.1.2019.