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Geometry I - WS 18

http://www.math.tu-berlin.de/geometrie/Lehre/WS18/GeometryI/

Tutorial from 14./15.11.

In order to define the hyperboloid model of hyperbolic geometry, we consider $\mathbb{R}^{n+1}$ together with a different scalar product. Before getting into definitions, one should stop and think for a moment about what changes, and what doesn't, if the space is the same but the scalar product is different. In the following we will denote the euclidean scalar product by $\langle \cdot, \cdot \rangle_{eucl}$.

Changes

Everything that is defined through the scalar product will change. These include

- Perception of angles, in particular perpendicularity: If α is the angle between two (non-zero) vectors a, b then α can be found via

$$\alpha = \arccos \left(\frac{\langle a, b \rangle_{eucl}}{\|a\| \, \|b\|} \right).$$

  By changing the scalar product and using the same formula, we can easily get things previously unheard of, for example a non-zero vector being perpendicular to itself. Be aware that orthogonality also appears in the definition of *orthonormal basis*.

- Well-definition of norm and distance: Every positive definite scalar product defines a norm ($\|a\| := \sqrt{\langle a, a \rangle}$), and every norm defines a metric ($d(x, y) := \|x - y\|$). If the new scalar product is not positive definite, these notions don't have to be well-defined anymore. How do we measure distance then?

- Topology: Metric defines the topology on the space, that is, tells us which sets are open and which are closed. Depending on this is for example the notion of continuity of maps(functions). A map is continuous if and only if preimages of open sets are open.

Task: Have a look in your lecture notes: how do we define angles, perpendicularity and distance in $\mathbb{R}^{n,1}$?

Things that stay the same

Things that don't change are those independent of the scalar product. These include for example the vector space properties of $\mathbb{R}^n$, such as linear independence, existence of a basis, dimension... BUT! Be aware of the following:

Recall the definitions of a plane in $\mathbb{R}^3$:

(a) $\Pi = \{v + \alpha b_1 + \beta b_2 \mid \alpha, \beta \in \mathbb{R}\}, v, b_1, b_2 \in \mathbb{R}^3$,

(b) $\Pi = \{x \in \mathbb{R}^3 \mid \langle x, c \rangle = d\}, d \in \mathbb{R}, c \in S^2$.

If you change the scalar product on $\mathbb{R}^3$, the first definition will still define a plane as we know it. The second, however, *might* yield utterly different spaces.

Bilinear and quadratic forms

Let V be a vector space over field K. Recall the following notions:

- A *form* on V is in general a map from V (or $V \times \dots V$) to K.
- A *linear* form on V is a map $f : V \to K$ s.t.

$$f(\lambda x + y) = \lambda f(x) + f(y)$$

for all $\lambda \in K, x, y \in V$.

- A *k-multilinear* form on V is a map $g : \underbrace{V \times \dots V}_{k-times} \to K$ that is linear in every component,

$$g(\dots, \underbrace{\lambda x + y}_{i-th\ coordinate}, \dots) = \lambda g(\dots, \underbrace{x}_{i-th\ coordinate}, \dots) + g(\dots, \underbrace{y}_{i-th\ coordinate}, \dots)$$

for all $\lambda \in K, x, y \in V$.
Question: What is your favourite multilinear form on vector spaces?

- A *bilinear* form is a multilinear form with $k = 2$.
- A *quadratic* form is a map $q : V \to K$ s.t.

$$q(\lambda x) = \lambda^2 q(x)$$

for all $\lambda \in K, x \in V$. $\lambda^2 = \lambda \cdot \lambda$, where $\cdot$ is the field multiplication.
Question: Does $q(x + y) = q(x) + q(y)$ still hold?

One can make **any bilinear form into a quadratic one**: Let $b : V \times V \to K$ be a bilinear form, define $q : V \to K$ via

$$q(v) := b(v, v).$$

However, the inverse works only partially. One can make any quadratic form into a **symmetric bilinear one**, and only under a certain condition on the field (K).

Questions: What does 'symmetric' mean? What is the formula for a definition of a symmetric bilinear form from a quadratic one? What is the condition on the field and why is it important?

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Under *scalar product* we understand a *non-degenerate symmetric bilinear form*. A symmetric bilinear form is non-degenerate, if its kernal consists of the zero vector only.

Question: Check how a kernal is defined for multilinear matrices.

Alternatively, one can say that a symmetric bilinear form is degenerate if its signature is of form $(n_+, n_-, n_0) = (k, n - k, 0)$. The signature come from the following theorem:

Theorem 1 *(Sylvester's law of inertia, Trägheitsatz) For a symmetric matric $A \in K^{n \times n}$ there exists a matrix $S \in Gl(n, K)$ such that*

$$SAS^T = diag(1, \dots, 1, -1, \dots, -1, 0, \dots, 0).$$

The numbers n_+ of 1's, n_- of -1's and n_0 of zeros are independent of S and the triple is called the signature (n_+, n_-, n_0) of A.

What does a symmetric bilinear form have to do with a symmetric matrix?
By choosing a basis $v_1, \ldots, v_n$ on V we create an isomorphism between V and K^n.

Question: How is this isomorphism defined?

Given a quadratic form b on V, define

$$A = [b(v_i, v_j)]_{i,j=1,\ldots,n}.$$

Then A is symmetric if and only if b is symmetric, and the bilinear form correspond via the isomorphism to the following multiplication:
Let $x, y \in V, x = \sum_{i=1}^{n} \lambda_i v_i, y = \sum_{i=1}^{n} \mu_i v_i$, then

$$b(x, y) = [\lambda_1, \ldots, \lambda_n] A [\mu_1, \ldots, \mu_n]^T.$$

If you change a basis on V, the matrix A changes, as well as the coefficients λ_i, μ_i. The equation above still holds, nevertheless. In fact, the matrix S from Sylvester's theorem represents exactly such choice of basis.

The following two definitions might come in handy for the homework:

- $v, w \in V$ are *orthogonal with respect to* b if $b(x, y) = 0$.
- A linear map $T : V \to V$ is *orthogonal transformation with respect to* b if $b(Tv, Tw) = b(v, w)$ for all $v, w \in V$.