Technische Universität Berlin

Fakultät II – Institut für Mathematik
Springborn, Kouřimská

!!Trial exam!!

Written Examination
Geometry I

Surname: .... First name: ....

Matr.–Nr.: .... Studiengang: ....

Apart from pens and rulers you are allowed to bring with you **one hand-written sheet of paper**, DIN A4.

Solutions written with a pencil **will not** be corrected.

Your final answers should fit into the framed boxes. After every exercise there is empty space for your calculations and notes. Additional paper will be provided upon request.

You may rip off the last two sheets, which contain a summary of all problems. Please leave the other sheets stapled together.

You have **90 minutes** for your work.

The total amount of points for this exam is 40.

1	2	3	4	Σ	Note

1. Question

(2+2+6)= 10 points

Let $A, B, C, D \in S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$. Two spherical triangles with vertices A, B, C and A, B, D are colunar if they share an edge and their other two edges belong to the same great circles.

a) Triangles with vertices A, B, C and A, B, D are colunar, and $C = (0, 0, 1)$. Determine D.

b) How many triangles are colunar to a given spherical triangle?

c) Let γ be the parametrization of the shorter great circle arc connecting

$$A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Find an explicit formula for γ and show that its length satisfies

$$\cos(L(\gamma)) = \langle A, B \rangle.$$

Overview of results:

a) $D = \Big(\qquad\qquad \Big).$

b) The number of triangles colunar to a given spherical triangle: ....

c)

$$\gamma : \Big[\qquad\qquad \Big] \to S^2, \qquad \gamma(t) =$$

Your work:

Notes and calculations:

2. Question

(4+2+4)= 10 points

Let $H^2 = \{x \in \mathbb{R}^3 \mid \langle x, x\rangle_{2,1} = -1\}$ be the hyperboloid model of the hyperbolic plane, ℓ_1 and ℓ_2 be two hyperbolic lines and $h_i = \{y \in H^2 \mid \langle y, n_i\rangle_{2,1} \geq 0\}, i \in \{1, 2\}$, be halfplanes bounded by ℓ_1 and ℓ_2. A hyperbolic rotation of H^2 with centre $x \in H^2$ is a map $T \in SO^+(2, 1)$ with $T(x) = x$.

Let

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad n_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} \sqrt{3} \\ -1 \\ 1 \end{pmatrix}.$$

a) A hyperbolic rotation $T \in SO^+(2, 1)$ with centre x maps h_1 to h_2. Find T. Show your work.

Let $\mathcal{C}$ be a Euclidean circle with radius $0 < r < 1$ and centre at the origin in the open unit disc D. Interpreting D as the Poincaré disc model makes $\mathcal{C}$ a hyperbolic circle.

b) Parametrize $\mathcal{C}$ by a curve γ.

c) The Riemannian metric on the Poincaré disc model is

$$g_p(v, w) = \frac{4}{(1 - p_1^2 - p_2^2)^2} \langle v, w\rangle_{\mathbb{R}^2}, \qquad p = (p_1, p_2), v, w \in D.$$

Use the Riemannian metric on D to calculate the length of γ as a curve in the Poincaré disc model.

Overview of results:

a) $T =$

Your work:

b) $\gamma : \left[\qquad\qquad \right] \to \mathcal{C}, \qquad \gamma(t) =$

c) .

Notes and calculations:

3. Question

(3+1+3+4)= 11 points

$$q : \mathbb{R}^3 \to \mathbb{R}, \qquad q(x) = x_1^2 + 2x_1x_2 + x_1x_3 - 7x_3^2.$$

a) Find the symmetric bilinear form $b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, (x, y) \mapsto b(x, y)$ such that $q(x) = b(x, x)$.

b) What is the signature of q?

c) Is q non-degenerate? Show your work.

d) Let e_1, e_2, e_3 denote the canonical basis of $\mathbb{R}^3$. Let $U = \operatorname{span}\{e_1, e_1 + e_2\}$. Find $U^\perp$ with respect to b.

Overview of results:

a) $b\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) =$

b) The signature of q is ____________________

c) Is q degenerate? ____________________

Your work:

d) $U^\perp =$

Notes and calculations:

4. Question

(9)= 9 points

Decide if the following statements are true or false. Briefly justify your answer if requested.

1. **(1 point)** The side lengths of the polar triangle are the interior angles of the original triangle and vice versa. (Justify your work.)

   True ☐ False ☐

2. **(1 point)** The great circles connecting $(1,0,0)$ and $(-1,0,0)$ in the unit sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ correspond to lines through the origin under the stereographic projection. (Justify your work.)

   True ☐ False ☐

3. **(1 point)** A symmetric bilinear form on a vector space V is non-degenerate if and only if the set $\{v \in V \mid b(v,v) = 0\} = \{0\}$. (Justify your work.)

   True ☐ False ☐

4. **(1 point)** Let $\mathbb{L} \subseteq \mathbb{R}^{2,1}$ be the light cone. Then $\{[x] \in \mathbb{R}^3 \mid x \in \mathbb{L}\}$ is a conic. (Justify your work.)

True ☐ False ☐

5. **(1 point)** A hyperbolic line in the hyperboloid model of the hyperbolic space $H^n \subseteq \mathbb{R}^{n,1}$ is the shortest path in the Lorentz space $\mathbb{R}^{n,1}$ connecting two vectors $x, y \in H^n$. (Justify your work.)

True ☐ False ☐

6. **(1 point)** The Klein-Beltrami model of hyperbolic geometry is conformal.

True ☐ False ☐

7. **(1 point)** The projective linear group $PGL(n, \mathbb{R})$ is a projective space.(Justify your work.)

True ☐ False ☐

True False

8. **(1 point)** Let $\ell_1, ..., \ell_4$ be four lines in a projective plane Π intersecting in a point P. Let $a, b \subseteq \Pi$ be two lines not containing P. Then

$$\mathrm{cr}(a \cap \ell_1, a \cap \ell_2, a \cap \ell_3, a \cap \ell_4) = \mathrm{cr}(b \cap \ell_1, b \cap \ell_2, b \cap \ell_3, b \cap \ell_4).$$

(Justify your work.)

True False

9. **(1 point)** For any five points in $\mathbb{R}P^2$, there is a unique conic section containing them. (Justify your work.)

Notes and calculations:

Overview of the exercises

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(Justify your work.)

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(Justify your work.)

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